대구과학고
김성일

< 조합등식 >

I. 이항계수의 기본등식

$(1+x)^n$ 의 전개식에서 각 항의 계수를 $\binom{n}{0}$ $\binom{n}{1}$ $\cdots$ $\binom{n}{n}$ 이라고 하면

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n \quad \cdots (*)$$

이 된다.

이때 $\binom{n}{k}$ 를 이항계수라고 부르고, 이 값은 조합의 수 ${}_nC_k$ 와 같다.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \quad \cdots ①$$

n 개에서 k 개를 뽑는 방법 수

$*$ 에 $x=-1,\ 1$ 을 대입하면

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} \cdots + (-1)^n\binom{n}{n} = 0$$

의 두 등식이 얻어진다.

$*$, ① 식을 이용하여 정리하면 다음의 기본등식 6가지를 얻을 수 있다. (암기)

(1) $\binom{n}{k} = \binom{n}{n-k}$

(2) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$: 파스칼의 공식

(3) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

(4) $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} = \binom{n}{k-m}\binom{n-k+m}{m}$ $(m \le k \le n)$

(5) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$

(6) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} \cdots + (-1)^n\binom{n}{n} = 0$

(주의) $k > n$ 이면 $\binom{n}{k} = 0$ 으로 약속한다. (n, k : 자연수, 0)

기본등식 증명에 계산식 ① 또는 $*$ 을 사용해도 된다. 그러나 몇몇은 조합적인 방법으로도 쉽게 증명되고 이해하기 쉬운 것이 있다.

예를 들어 (4) 를 생각하자.

— Next page

예제 1) 기본등식 4를 증명하시오.

(풀이) n명에서 k명을 뽑고, 그중 m명의 대표위원을 뽑는 경우를 생각하자.

n명에서 k명을 먼저 뽑고, 그 중 m명을 뽑는 수 $\binom{n}{k}\binom{k}{m}$

n명에서 m명을 미리 뽑고, $n-m$명에서 $k-m$명을 뽑으면 $\binom{n}{m}\binom{n-m}{k-m}$

n명에서 대표위원 아닌 $k-m$명을 먼저 뽑고, 나머지에서 m명을 뽑으면

$\binom{n}{k-m}\binom{n-k+m}{m}$

이것은 모두 같아야 하므로

$$\binom{n}{k}\binom{k}{m}=\binom{n}{m}\binom{n-m}{k-m}=\binom{n}{k-m}\binom{n-k+m}{m}$$

문제 1) 기본등식 (1)(2)(3)(4)를 조합적인 방법과 정의를 이용한 계산으로 보이는 방법 두 가지로 증명해보라.

예제 2) $S_{n,m}=\sum_{k=0}^{m}(-1)^k\binom{n}{k}$ 를 계산하시오. $(m\le n)$

(풀이) $m=n$ 인 경우 기본등식 (6)에 의해 $S_{n,m}=0$

$m<n$ 인 경우 기본등식 (2)에 의해

$$\begin{aligned}
S_{n,m} &= 1+\sum_{k=1}^{m}(-1)^k\left\{\binom{n-1}{k}+\binom{n-1}{k-1}\right\}\\
&= 1+\sum_{k=1}^{m}(-1)^k\binom{n-1}{k}+\sum_{k=1}^{m}(-1)^k\binom{n-1}{k-1}\\
&= 1+\sum_{k=2}^{m+1}(-1)^{k-1}\binom{n-1}{k-1}+\sum_{k=1}^{m}(-1)^k\binom{n-1}{k-1}\\
&= 1-\binom{n-1}{0}+\sum_{k=2}^{m}\{(-1)^k+(-1)^{k-1}\}\binom{n-1}{k-1}+(-1)^m\binom{n-1}{m}\\
&= (-1)^m\binom{n-1}{m}
\end{aligned}$$

따라서

$$S_{n,m}=\begin{cases}0 & (m=n)\\ (-1)^m\binom{n-1}{m} & (m<n)\end{cases}$$

(증명끝)

예제 3) $\sum_{k=0}^{n}\binom{2n}{k}=2^{2n-1}+\frac{1}{2}\binom{2n}{n}$ 을 증명하시오.

풀이) $S_n = \sum_{k=0}^{n} \binom{2n}{k}$ 라 두자.

$\binom{2n}{k} = \binom{2n}{2n-k}$ 이므로

$$S_n = \sum_{k=n}^{2n} \binom{2n}{k}$$

$$2S_n = \sum_{k=0}^{n} \binom{2n}{k} + \sum_{k=n}^{2n} \binom{2n}{k}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} + \binom{2n}{n}$$

$$= 2^{2n} + \binom{2n}{n}$$

$$S_n = 2^{2n-1} + \frac{1}{2}\binom{2n}{n}$$

(증명끝)

문제 2) $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$ 임을 증명하라

문제 3) $\sum_{k=0}^{n-1} \binom{2n-1}{k} = 2^{2n-2}$ 임을 증명하라.

예제 4) $P_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$, $g_n = \sum_{k=0}^{n} (-1)^k \frac{1}{k+1}\binom{n}{k}$ 를 계산하시오.

풀이) 기본등식 (4) 에 의하여

$$\frac{n+1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}$$

$$P_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$$

$$= \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1}$$

$$= \frac{2^{n+1}-1}{n+1} \quad (\because \text{기본등식 } 5)$$

$$g_n = \sum_{k=0}^{n} (-1)^k \frac{1}{k+1}\binom{n}{k}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{1}{n+1}\binom{n+1}{k+1}$$

$$= \frac{+\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} + \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k}}{n+1}$$

$$= \frac{(-1)^0 \binom{n+1}{0}}{n+1} = \frac{1}{n+1}$$

(증명끝)

문제 4) $\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$ 임을 증명하시오.

조환등식

예제 5) $\sum_{k=1}^{n} k^2 \binom{n}{k}$ 를 계산하시오.

(풀이) 기본등식 (2)를 적용하면

$$\sum_{k=1}^{n} k^2\binom{n}{k} = n\sum_{k=1}^{n} k \binom{n-1}{k-1}$$
$$= n\sum_{k=1}^{n}(k-1)\binom{n-1}{k-1} + n\sum_{k=1}^{n}\binom{n-1}{k-1}$$
$$= n\sum_{k=2}^{n}(k-1)\binom{n-1}{k-1} + n\cdot 2^{n-1}$$
$$= n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2} + n\cdot 2^{n-1}$$
$$= n(n-1)\,2^{n-2} + n\cdot 2^{n-1}$$
$$= n(n+1)\,2^{n-2}$$

(풀이끝)

문제 5) $\sum_{k=0}^{n}(k+1)\binom{n}{k} = 2^{n-1}(n+2)$ 임을 증명하시오.

예제 6) $S_{m,n} = \sum_{k=m}^{n}(-1)^k \binom{n}{k}\binom{k}{m} = (-1)^m \delta_{mn}$ $(m \le n)$ 임을 증명하시오.

단 $\delta_{mn} = \begin{cases} 1 & (m=n) \\ 0 & (m \ne n) \end{cases}$

(풀이) $m=n$ 이면 $S_{m,n} = (-1)^n\binom{n}{n}\binom{n}{n} = (-1)^n$ ∴ 성립한다

$m \ne n$ 이면 $(m<n)$ 이면

기본등식 (4) 에 의하여

$$S_{m,n} = \sum_{k=m}^{n}(-1)^k\binom{n}{m}\binom{n-m}{k-m}$$
$$= \binom{n}{m}\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}$$

$k-m=i$ 라 치환하면

$$S_{m,n} = \binom{n}{m}\sum_{i=0}^{n-m}(-1)^{i+m}\binom{n-m}{i}$$
$$= 0 \quad (\because \text{기본등식 } (6))$$

(증명끝)

문제6) $\sum_{k=r}^{n}(-1)^k \frac{1}{k+1}\binom{n}{k}\binom{k}{r} = \frac{(-1)^r}{n+1}$ 임을 증명하시오.

문제 7) $\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}$ 임을 증명하시오. $(m \le n)$

예제 7) $\sum_{k=0}^{m}(-1)^k\binom{n}{k}\frac{m}{m+k} = \frac{1}{\binom{m+n}{m}}$ 임을 증명하시오.

(해설) $b_n =$ (좌변) 이라 하여
점화식을 구하고 일반항을 추정하는 것이 쉽다.
기본등식 만 가지고는 이항계수의 역수꼴이 나오기 어렵다

(풀이) $m=1$ 이면 예제 3의 g_n과 같다.
증명하고자 하는 등식의 좌변을 b_n라 두면

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{m}{m+k}$$

$$= 1 + \sum_{k=1}^{n-1} (-1)^k \left\{ \binom{n-1}{k} + \binom{n-1}{k-1} \right\} \frac{m}{m+k} + (-1)^n \binom{n}{n} \frac{m}{m+n}$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{m}{m+k} + \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{m}{m+k}$$

$$= b_{n-1} + \frac{m}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k}{m+k}$$

$$= b_{n-1} + \frac{m}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{m+k-m}{m+k} \right)$$

$$= b_{n-1} + \frac{m}{n} \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{m}{m+k} \right)$$

$$= b_{n-1} + \frac{m}{n} (-b_n)$$

따라서 $b_n = \frac{n}{m+n} b_{n-1}$

$b_0 = 1$ 이므로

$$b_n = \frac{n}{m+n} \cdot \frac{(n-1)}{m+(n-1)} b_{n-2}$$

$$= \cdots = \frac{n!\, m!}{(n+m)!} b_0 = \frac{1}{\binom{m+n}{m}}$$ (증명끝)

문제 8) $\sum_{k=0}^{n} (-1)^k \, k \binom{n}{k}$ 값을 구하시오. $(n > 1)$

예제 8) $\sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k} \binom{n}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ 임을 증명하시오.

풀이) $n=1$ 이면 성립한다.
$n=m$ 일때 성립한다고 가정하자.

$$\sum_{k=1}^{m+1} (-1)^{k+1} \frac{1}{k} \binom{m+1}{k}$$

$$= \sum_{k=1}^{m} (-1)^{k+1} \frac{1}{k} \left(\binom{m}{k} + \binom{m}{k-1} \right) + (-1)^{m+2} \frac{1}{m+1}$$

$$= \sum_{k=1}^{m} (-1)^{k+1} \frac{1}{k} \binom{m}{k} + \sum_{k=1}^{m+1} (-1)^{k+1} \frac{1}{k} \binom{m}{k-1}$$

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{m} + \frac{1}{m+1} \sum_{k=1}^{m+1} (-1)^{k+1} \binom{m+1}{k}$$

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{m} + \frac{1}{m+1} \left(\sum_{k=0}^{m+1} (-1)^{k+1} \binom{m+1}{k} - (-1)^1 \right)$$

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{m+1}$$

즉 $n=m+1$ 일때도 성립한다. (∵ 수학적 귀납법) (증명끝)

문제 9) $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}x^k = \frac{(1+x)^{n+1}-1}{(n+1)x}$ 임을 증명하시오.

예제 9) $\sum_{k=0}^{n} (-1)^k 2^{2n-2k}\binom{2n-k+1}{k} = n+1$ 임을 증명하시오.

(풀이) 위식의 좌변을 a_n 라 하자.

$a_n = \sum_{k=0}^{n}(-1)^k 2^{2n-2k}\binom{2n-k+1}{k}$

$= 2^{2n} + \sum_{k=1}^{n}(-1)^k 2^{2n-2k}\left(\binom{2n-k}{k}+\binom{2n-k}{k-1}\right)$ ∵ 기본등식 (2)

$= \sum_{k=0}^{n}(-1)^k 2^{2n-2k}\binom{2n-k}{k} + \sum_{k=1}^{n}(-1)^k 2^{2n-2k}\binom{2n-k}{k-1}$

$= \sum_{k=0}^{n}(-1)^k 2^{2n-2k}\binom{2n-k}{k} + \sum_{k=0}^{n-1}(-1)^{k+1} 2^{2n-2(k+1)}\binom{2n-(k+1)}{k}$

$= \sum_{k=0}^{n}(-1)^k 2^{2n-2k}\binom{2n-k}{k} + \sum_{k=0}^{n-1}(-1)^{k+1} 2^{2(n-1)-2k}\binom{2(n-1)-k+1}{k}$

$= \sum_{k=0}^{n}(-1)^k 2^{2n-2k}\binom{2n-k}{k} + (-a_{n-1})$

$b_n = \sum_{k=0}^{n}(-1)^k 2^{2n-2k}\binom{2n-k}{k}$ 라 두자

$b_n = a_n + a_{n-1}$ ... ①

b_n도 a_n처럼 풀어보자.

$b_n = \sum_{k=0}^{n}(-1)^k 2^{2n-2k}\binom{2n-k}{k}$

$= 2^{2n} + \sum_{k=1}^{n-1}(-1)^k 2^{2n-2k}\left(\binom{2n-k-1}{k}+\binom{2n-k-1}{k-1}\right) + (-1)^n$

$= \sum_{k=0}^{n-1}(-1)^k 2^{2n-2k}\binom{2n-k-1}{k} + \sum_{k=1}^{n}(-1)^k 2^{2n-2k}\binom{2n-k-1}{k-1}$

$= 4\sum_{k=0}^{n-1}(-1)^k 2^{2(n-1)-2k}\binom{2(n-1)-k+1}{k} + \sum_{k=0}^{n-1}(-1)^{k+1} 2^{2(n-1)-2k}\binom{2n-k}{k}$

$b_n = 4a_{n-1} - b_{n-1}$ ... ②

①을 ②에 대입하면

$a_n + a_{n-1} = 4a_{n-1} - (a_{n-1} + a_{n-2})$

$a_n = 2a_{n-1} - a_{n-2}$

$a_n - a_{n-1} = a_{n-1} - a_{n-2} = \cdots = a_2 - a_1 = a_1 - a_0$

$a_0 = 1,\ a_1 = 2$ 이므로

$a_n = n+1$ (증명끝)

Note 1) a_n을 직접계산하지 않고, b_n을 도입하여, 둘 사이 관계를 유도하여 해결할수 있다.

예제 10. $\sum_{k=0}^{2n-1}(-1)^{k+1}(k+1)\frac{1}{\binom{2n}{k}} = 0$ 임을 증명하시오.

(풀이) $y_n = \sum_{k=0}^{2n-1}(-1)^{k+1}\frac{1}{\frac{1}{k+1}\binom{2n}{k}}$

$= \sum_{k=0}^{2n-1}(-1)^{k+1}\frac{2n+1}{\binom{2n+1}{k+1}}$

$= \sum_{k=0}^{2n-1}(-1)^{k+1}\frac{2n+1}{\binom{2n+1}{2n-k}} = \sum_{k=0}^{2n-1}(-1)^{k+1}\frac{2n-k}{\binom{2n}{2n-k-1}}$

$$= \sum_{i=0}^{2n-1} (-1)^{(2n-1-i)+1} \frac{2n-(2n-1-i)}{\binom{2n}{2n-1-(2n-1-i)}}$$

$$= \sum_{i=0}^{2n-1} (-1)^{2n-i} \frac{i+1}{\binom{2n}{i}}$$

$$= \sum_{i=0}^{2n-1} (-1)^{i}(i+1) \frac{1}{\binom{2n}{i}} = -y_n$$

∴ $y_n = -y_n$ $\quad y_n = 0$ (증명끝)

문제 (0) $\sum_{k=1}^{2n-1} (-1)^{k-1}(n-k)\frac{1}{\binom{2n}{k}} = 0$ 임을 증명하라.

예제 (1) $\sum_{k=1}^{2n-1}(-1)^{k-1}\frac{1}{\binom{2n}{k}} = \frac{1}{n+1}$ 임을 증명하라

(풀이) 1. $a_n = \sum_{k=1}^{2n-1}(-1)^{k-1}\frac{1}{\binom{2n}{k}}$, $b_n = \sum_{k=1}^{2n-1}(-1)^{k-1} k \frac{1}{\binom{2n}{k}}$ 라고 하자.

예제 (0에 의하여

$$a_n+b_n = \sum_{k=1}^{2n-1}(-1)^{k-1}(k+1)\frac{1}{\binom{2n}{k}} = 0-(-1)^1 = 1 \quad \cdots ①$$

문제 (0에 의하여

$$na_n - b_n = \sum_{k=1}^{2n-1}(-1)^{k-1}(n-k)\frac{1}{\binom{2n}{k}} = 0 \quad \cdots ②$$

② ①에서 $(n+1)a_n = 1$ $\quad a_n = \frac{1}{n+1}$

$b_n = \frac{n}{n+1}$

2. 직접 계산하면 다음 등식이 얻어진다

$$\frac{2n+2}{2n+1}\frac{1}{\binom{2n}{k}} = \frac{1}{\binom{2n+1}{k}} + \frac{1}{\binom{2n+1}{k+1}} \quad \cdots ⊗$$

따라서

$$\frac{2n+2}{2n+1}\sum_{k=1}^{2n}\frac{1}{\binom{2n}{k}}(-1)^{k-1}$$

$$= \sum_{k=1}^{2n}\left(\frac{1}{\binom{2n+1}{k}}+\frac{1}{\binom{2n+1}{k+1}}\right)(-1)^{k-1}$$

$$= -\sum_{k=1}^{2n}(-1)^k\frac{1}{\binom{2n+1}{k}} + \sum_{k=2}^{2n+1}(-1)^k\frac{1}{\binom{2n+1}{k}}$$

$$= (-1)^{2n+1} + \sum_{k=2}^{2n}\left((-1)^k-(-1)^k\right)\frac{1}{\binom{2n+1}{k}} + \frac{1}{\binom{2n+1}{1}}$$

$$= \frac{1}{2n+1} - 1 = \frac{-2n}{2n+1}$$

∴ $\sum_{k=1}^{2n}\frac{1}{\binom{2n}{k}}(-1)^{k-1} = \frac{-2n}{2n+2} = \frac{-n}{n+1}$

$$\sum_{k=1}^{2n-1}\frac{1}{\binom{2n}{k}}(-1)^{k-1} = -\frac{n}{n+1} - (-1)^{2n-1}\frac{1}{\binom{2n}{2n}}$$

$$= \frac{1}{n+1}$$

(해설) ⊗ 부분 : $\frac{2n+1}{k+1}\binom{2n}{k} = \binom{2n+1}{k+1}$ $\cdots ②$

$\frac{2n+2}{k+1}\binom{2n+1}{k} = \binom{2n+2}{k+1} = \binom{2n+1}{k+1}+\binom{2n+1}{k}$ $\cdots ①$

①/② $\quad \frac{2n+2}{2n+1}\frac{1}{\binom{2n}{k}} = \frac{\binom{2n+1}{k+1}+\binom{2n+1}{k}}{\binom{2n+1}{k}\binom{2n+1}{k+1}}$

문제 (1) $\sum_{k=0}^{n}(-1)^k \frac{1}{\binom{n}{k}} = \frac{n+1}{n+2}\left(1+(-1)^n\right)$ 임을 증명하시오.

예제 (2) $\sum_{k=1}^{n-1} \frac{1}{k(n-k)} \binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1} = \frac{1}{n}\binom{2n-2}{n-1}$ 임을 증명하시오.
$(n=2,3,4,\cdots)$

(풀이)
$$\sum_{k=1}^{n-1} \frac{n}{k(n-k)} \binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1}$$
$$= \sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right)\binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1}$$
$$= \sum_{k=1}^{n-1}\frac{1}{k}\binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1} + \sum_{k=1}^{n-1}\frac{1}{n-k}\binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1}$$
$$= \sum_{k=1}^{n-1}\frac{1}{k}\binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1} + \sum_{k=1}^{n-1}\frac{1}{k}\binom{2n-2k-2}{n-k-1}\binom{2k-2}{k-1}$$
$$= 2\sum_{k=1}^{n-1}\frac{1}{k}\binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1}$$

즉 다음을 증명하면 된다
$$\sum_{k=1}^{n-1}\frac{1}{k}\binom{2k-2}{k-1}\binom{2n-2k-2}{n-k-1} = \frac{1}{2}\binom{2n-2}{n-1} \quad \cdots *$$

수학적 귀납법으로 증명해보자.

i) $n=2$ 이면 $*$는 성립한다.

ii) $n=m$ 일때 성립가정.

$n=m+1$ 이면
$$\sum_{k=1}^{m}\frac{1}{k}\binom{2k-2}{k-1}\binom{2m-2k}{m-k}$$
$$= \sum_{k=1}^{m-1}\frac{1}{k}\binom{2k-2}{k-1}\binom{2m-2k}{m-k} + \frac{1}{m}\binom{2m-2}{m-1}$$
$$= \sum_{k=1}^{m-1}\frac{1}{k}\cdot\binom{2k-2}{k-1}\frac{2m-2k}{m-k}\binom{2m-2k-1}{m-k-1} + \frac{1}{m}\binom{2m-2}{m-1}$$
$$= \sum_{k=1}^{m-1}\frac{2}{k}\binom{2k-2}{k-1}\binom{2m-2k-1}{m-k} + \frac{1}{m}\binom{2m-2}{m-1}$$
$$= \sum_{k=1}^{m-1}\frac{2}{k}\binom{2k-2}{k-1}\frac{2(m-k)-1}{m-k}\binom{2m-2k-2}{m-k-1} + \frac{1}{m}\binom{2m-2}{m-1}$$
$$= \sum_{k=1}^{m-1}\frac{2}{k}\left(2-\frac{1}{m-k}\right)\binom{2k-2}{k-1}\binom{2m-2k-2}{m-k-1} + \frac{1}{m}\binom{2m-2}{m-1}$$
$$= 4\sum_{k=1}^{m-1}\frac{1}{k}\binom{2k-2}{k-1}\binom{2m-2k-2}{m-k-1} - 2\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\binom{2k-2}{k-1}\binom{2m-2k-2}{m-k-1}$$
$$+\frac{1}{m}\binom{2m-2}{m-1}$$
$$= 4\cdot\frac{1}{2}\binom{2m-2}{m-1} - \frac{4}{m}\sum_{k=1}^{m-1}\frac{1}{k}\binom{2k-2}{k-1}\binom{2m-2k-2}{m-k-1} + \frac{1}{m}\binom{2m-2}{m-1}$$
$$= 2\binom{2m-2}{m-1} - \frac{4}{2m}\binom{2m-2}{m-1} + \frac{1}{m}\binom{2m-2}{m-1}$$
$$= \left(2-\frac{2}{m}+\cancel{m}\right)\binom{2m-2}{m-1}$$
$$= \frac{2m-1}{m}\binom{2m-2}{m-1}$$
$$= \binom{2m-1}{m}$$
$$= \binom{2m-1}{m-1}$$
$$= \frac{1}{2}\cdot\frac{2m}{m}\binom{2m-1}{m-1}$$
$$= \frac{1}{2}\binom{2m}{m}$$
즉 $n=m+1$ 에서도 성립한다. (증명끝)

문제 (2) n이 짝수이면 $\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1}=2^{n-1}$
n이 홀수이면 $\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n}=2^{n-1}$ 임을 증명하라.

문제 (3) $\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^2}\binom{n}{k} = \frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)$ 을 증명하라.

조합등식

문제 14) $\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}\binom{n}{k}\{1-(1-x)^k\}=x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}$ 를 증명하라

문제 15) $\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}=\frac{2^{2n}}{2n+1}\binom{2n}{n}^{-1}$ 을 증명하라.

문제 16) $\sum_{k=0}^{n}\frac{(-1)^k}{1-2k}\binom{n}{k}=2^{2n}\binom{2n}{n}^{-1}$ 을 증명하라.

문제 17) $\sum_{k=0}^{n}k\binom{2n+1}{k}=(2n+1)2^{2n-1}-\frac{2n-1}{2}\binom{2n}{n}$

문제 18) $\sum_{k=0}^{n}k\binom{2n}{n-k}=n\cdot\binom{2n-1}{n}$

문제 19) $\sum_{k=0}^{n}k\binom{2n+1}{n-k}=(2n+1)\binom{2n-1}{n}-2^{2n-1}$

문제 20) $\sum_{k=0}^{n}\binom{2n+1}{2k+1}x^k=\frac{(1+\sqrt{x})^{2n+1}-(1-\sqrt{x})^{2n+1}}{2\sqrt{x}}$

문제 21) $\sum_{k=r}^{n}\binom{n}{k}\binom{k}{r}x^k(1-x)^{n-k}=\binom{n}{r}x^r$