Certifying Large Branch-width

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Abstract

Branch-width is defined for graphs, matroids, and, more generally, arbitrary symmetric submodular functions. For a finite set V, a function f on the set of subsets 2^V of V is *submodular* if $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$, and *symmetric* if $f(X) = f(V \setminus X)$. We discuss the computational complexity of recognizing that symmetric submodular functions have branch-width at most k for fixed k. An integer-valued symmetric submodular function f on 2^V is a *connectivity function* if $f(\emptyset) = 0$ and $f(\{v\}) \leq 1$ for all $v \in V$. We show that for each constant k, if a connectivity function f on 2^V is presented by an oracle and the branch-width of f is larger than k, then there is a certificate of polynomial size (in $|V|$) such that a polynomial-time algorithm can verify the claim that branch-width of f is larger than k. In particular it is in coNP to recognize matroids represented over a fixed field with branch-width at most k for fixed k.

1 Introduction

Branch-width (for graphs) was defined by Robertson and Seymour [6]. We will define the more general *branch-width* of *symmetric submodular* functions later in Section 2. One natural question is the following.

> Let k be a fixed constant and let V be a finite set. What is the time complexity of deciding whether the branch-width of a symmetric submodular function $f : 2^V \to \mathbb{Z}$ is at most k?
>
> (We assume that f is presented by an oracle.)

We answer this question partially when $f(\emptyset) = 0$ and $f(\{v\}) \leq 1$ for all $v \in V$. In this case, we say that f is a *connectivity function.* Symmetric submodular functions defining branch-width of matroids [6] and rank-width of graphs [5] are instances of connectivity functions. We show that if the branch-width of a connectivity function is larger than k, then there is a certificate of this fact, of polynomial size in $|V|$, which can be checked in time a polynomial in $|V|$.

We are not yet able to find a polynomial-time algorithm to decide whether branch-width is at most k, but in [5], we give a polynomial-time "approximation" algorithm that, for fixed k, either confirms that branch-width of a connectivity function is larger than k or obtains a branch-decomposition of width at most $3k+1$.

There have been answers for our problem for a few special symmetric submodular functions separately. We summarize them in Table 1. In particular, it is open whether there exists a polynomial-time algorithm that decides whether a matroid (given by an independence oracle) has branch-width at most k for fixed k. Moreover, this problem is open when the input matroid is represented over a fixed non-finite field. Our result implies that it is in NP∩coNP to decide that branch-width of represented matroids is at most k; in this case we do not need an oracle to obtain the input matroid and therefore we can say that our algorithm is in coNP.

Object	Results
Branch-width of graphs G	Linear time [1]
Branch-width of matroids $\mathcal{M}$ represented over a fixed finite field	$O(\lvert E(\mathcal{M})\rvert^3)$-time[1] [2]
Rank-width of graphs G	$O(\lvert V(G)\rvert^3)$-time [4]
Branch-width of matroids	?

Table 1: Parametrized algorithms on deciding branch-width $\leq k$ for fixed k

2 Definitions

Let us write $\mathbb{Z}$ to denote the set of integers. Let V be a finite set and $f : 2^V \to \mathbb{Z}$ be a function. If

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

for all $X, Y \subseteq V$, then f is said to be *submodular.* If f satisfies $f(X) = f(V \setminus X)$ for all $X \subseteq V$, then f is said to be *symmetric.*

A *subcubic tree* is a tree with at least two vertices such that every vertex is incident with at most three edges. A *leaf* of a tree is a vertex incident with exactly one edge. We call $(T, \mathcal{L})$ a *branch-decomposition* of a

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[1] The input is given by the matrix representation of matroids.

symmetric submodular function f if T is a subcubic tree and $\mathcal{L} : V \to \{t : t \text{ is a leaf of } T\}$ is a bijective function. (If $|V| \le 1$ then f admits no branch-decomposition.)

For an edge e of T, the connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a branch-decomposition $(T, \mathcal{L})$ is $f(\mathcal{L}^{-1}(X))$. The *width* of $(T, \mathcal{L})$ is the maximum width of all edges of T. The *branch-width* of f is the minimum width of a branch-decomposition of f. (If $|V| \le 1$, we define that the branch-width of f is $f(\emptyset)$.)

3 Comparing branch-width with a fixed number

Let f be a symmetric submodular functions on 2^V. To prove that branch-width of f is at most k, we can provide a natural certificate, a branch-decomposition of width at most k. However it is nontrivial to disprove that branch-width of f is at most k. We use the notion called a *tangle*, which is dual to the notion of branch-width and was introduced by Robertson and Seymour [6].

A class $\mathcal{T}$ of subsets of V is called a *f-tangle* of order k if it satisfies the following four axioms.

(T1) For all $A \in \mathcal{T}$, we have $f(A) < k$.

(T2) For all $A \subseteq V$, if $f(A) < k$, then either $A \in \mathcal{T}$ or $V \setminus A \in \mathcal{T}$.

(T3) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq V$.

(T4) For all $v \in V$, we have $V \setminus \{v\} \notin \mathcal{T}$.

Proposition 3.1. *Let $\mathcal{T}$ be a f-tangle of order k. If $A \in \mathcal{T}$, $B \subseteq A$, and $f(B) < k$, then $B \in \mathcal{T}$.*

Proof. By (T2), either $B \in \mathcal{T}$ or $V \setminus B \in \mathcal{T}$. Since $(V \setminus B) \cup A \cup A = V$, the f-tangle $\mathcal{T}$ cannot contain $V \setminus B$ by (T3). Hence $B \in \mathcal{T}$.

Robertson and Seymour [6] showed that tangles are related to branch-width.

Theorem 3.1. (Robertson and Seymour [6]) *There is no f-tangle of order $k + 1$ if and only if branch-width of f is at most k.*

Therefore to show that the branch-width of f is larger than k, it is enough to provide a f-tangle $\mathcal{T}$ of order $k + 1$. However, $\mathcal{T}$ might contain exponentially many subsets of V. So, we need to devise a way to encode a f-tangle of order $k+1$ into a certificate of polynomial size in $|V|$. If f is a connectivity function, then there is such an encoding as we explain later. We need the following lemmas. For disjoint subsets of X and Y, let $f_{\min}(X, Y) = \min_{X \subseteq U \subseteq V \setminus Y} f(U)$.

Lemma 3.1. *For a connectivity function f on subsets of V,*

$$f_{\min}(A, B) + f_{\min}(C, D) \ge f_{\min}(A \cap C, B \cup D) + f_{\min}(A \cup C, B \cap D).$$

Proof. Let S be a subset of V such that $A \subseteq S \subseteq V \setminus B$ and $f(S) = f_{\min}(A, B)$. Let T be a subset of V such that $C \subseteq T \subseteq V \setminus D$ and $f(T) = f_{\min}(C, D)$. By the submodularity of f, we deduce

$$f(S) + f(T) \ge f(S \cap T) + f(S \cup T)$$

and moreover $f(S \cap T) \ge f_{\min}(A \cap C, B \cup D)$ and $f(S \cup T) \ge f_{\min}(A \cup C, B \cap D)$.

Lemma 3.2. *For a connectivity function f on subsets of V,*

$$0 \le f_{\min}(A, B) \le \min(|A|, |B|).$$

Proof. Since f is symmetric, $f_{\min}(A, B) = f_{\min}(B, A)$ and therefore it is enough to show that $f_{\min}(A, B) \le |A|$. We proceed by induction on $|A|$. If $A = \emptyset$, then it is clear that $f_{\min}(\emptyset, B) \le 0$. Now let us assume that $v \in A$. Then by Lemma 3.1, $f_{\min}(A, B) \le f_{\min}(A \setminus \{v\}, B) + f_{\min}(\{v\}, B)$ and therefore $f_{\min}(A, B) \le |A|$.

Lemma 3.3. *Let f be a connectivity function on subsets of V. For a subset Z of V, there exist a subset X of Z and a subset Y of $V \setminus Z$ such that $f_{\min}(X, Y) = f(Z)$ and $|X| = |Y| = f(Z)$.*

Proof. Let X be the maximum subset of Z such that $f_{\min}(X, V \setminus Z) = |X|$. For all $v \in Z \setminus X$, $f_{\min}(X \cup \{v\}, V \setminus Z) \le |X| + 1$ by Lemma 3.2. Moreover $f_{\min}(X \cup \{v\}, V \setminus Z) \ge f_{\min}(X, V \setminus Z) = |X|$ by definition. Since X is chosen maximally, $f_{\min}(X \cup \{v\}, V \setminus Z) \neq |X| + 1$ and therefore $f_{\min}(X \cup \{v\}, V \setminus Z) = |X|$ for all $v \in Z \setminus X$. By Lemma 3.1, we deduce that $f_{\min}(Z, V \setminus Z) = |X|$ and therefore $|X| = f(Z)$.

We now take Y as a maximum subset of $V \setminus Z$ such that $f_{\min}(X, Y) = |Y|$. By the similar argument, we deduce that $f_{\min}(X, Y) = f(Z) = |X| = |Y|$.

For a connectivity function f on subsets of V, we say that (P, μ) is a *f-tangle kit* of order k if $P = \{(X, Y) : X, Y \subseteq V, X \cap Y = \emptyset, f_{\min}(X, Y) = |X| = |Y| < k\}$ and $\mu : P \to 2^V$ is a function satisfying the following three axioms.

(K1) $\mu(X_1, Y_1) \cup \mu(X_2, Y_2) \cup \mu(X_3, Y_3) \neq V$ for all $(X_i, Y_i) \in P$ for $i \in \{1, 2, 3\}$.

(K2) for all $(A, B) \in P$, there is no Z such that $A \subseteq Z \subseteq V \setminus B$, $f(Z) = |A|$, and $Z \not\subseteq \mu(A, B)$ and $V \setminus Z \not\subseteq \mu(B, A)$.

Equivalently for all $x \in V \setminus (\mu(A,B) \cup B)$ and $y \in V \setminus (\mu(B,A) \cup A)$, if $x \neq y$, then $f_{\min}(A \cup \{x\}, B \cup \{y\}) > |A|$.

(K3) $|\mu(X,Y)| \neq |V| - 1$ for all $(X,Y) \in P$.

In the following theorem we show that for a connectivity function f, f-tangle kits play the same role as f-tangles.

THEOREM 3.2. *Let f be a connectivity function on V. There exists a f-tangle of order k if and only if there exists a f-tangle kit of order k.*

Proof. Let $\mathcal{T}$ be a f-tangle of order k. We claim that there exists a f-tangle kit of order k. Let $P = \{(X,Y) : X, Y \subseteq V, X \cap Y = \emptyset, f_{\min}(X,Y) = |X| = |Y| < k\}$. We claim that for each $(X,Y) \in P$, there is a unique maximal set $Z \in \mathcal{T}$, denoted by $\mu(X,Y)$, such that $X \subseteq Z \subseteq V \setminus Y$ and $f(Z) = f_{\min}(X,Y)$. Suppose that Z_1 and Z_2 are contained in $\mathcal{T}$ and $X \subseteq Z_1 \subseteq V \setminus Y$, $X \subseteq Z_2 \subseteq V \setminus Y$, and $f(Z_1) = f(Z_2) = f_{\min}(X,Y)$. By submodularity,

$$f(Z_1 \cup Z_2) + f(Z_1 \cap Z_2) \leq f(Z_1) + f(Z_2) = 2f_{\min}(X,Y).$$

Since $f(Z_1 \cup Z_2) \geq f_{\min}(X,Y)$ and $f(Z_1 \cap Z_2) \geq f_{\min}(X,Y)$, we deduce that $f(Z_1 \cup Z_2) = f(Z_1 \cap Z_2) = f_{\min}(X,Y)$. Since $Z_1 \cup Z_2 \cup (V \setminus (Z_1 \cup Z_2)) = V$, we obtain that $Z_1 \cup Z_2 \in \mathcal{T}$. Thus $\mu : P \to 2^V$ is well-defined. (K1) follows (T3) and (K3) follows (T4). (K2) is true by (T2) and the construction of μ.

Conversely let us assume that we are given a f-tangle kit (P,μ) of order k. We construct a f-tangle $\mathcal{T}$ of order k as follows.

For all Z such that $f(Z) < k$, we choose $(A,B) \in P$ such that

$$|A| = |B| = f(Z) \text{ and } A \subseteq Z \subseteq V \setminus B.$$

If $Z \subseteq \mu(A,B)$, then $Z \in \mathcal{T}$. Otherwise, $V \setminus Z \in \mathcal{T}$.

Let us first show that this is well-defined. Let Z be a subset of V such that $f(Z) < k$. By Lemma 3.3, there are $A \subseteq Z$ and $B \subseteq V \setminus Z$ such that $f_{\min}(A,B) = |A| = |B| = f(Z)$. By (K2), either $Z \subseteq \mu(A,B)$ or $V \setminus Z \subseteq \mu(B,A)$. Suppose that there are two pairs $(A_1,B_1), (A_2,B_2) \in P$ such that $A_1, A_2 \subseteq Z$, $B_1, B_2 \subseteq V \setminus Z$, $f_{\min}(A_1,B_1) = f_{\min}(A_2,B_2) = f(Z)$, and $Z \subseteq \mu(A_1,B_1)$ but $Z \not\subseteq \mu(A_2,B_2)$. We obtain that $\mu(B_2,A_2) \cup \mu(A_1,B_1) = V$, because $V \setminus Z \subseteq \mu(B_2,A_2)$ by (K2). This contradicts (K1).

We now claim that the f-tangle axioms are satisfied by $\mathcal{T}$. Axioms (T1) and (T2) are true by construction. To show (T3), assume that $A_i \in \mathcal{T}$ for all $i \in 1,2,3$. There exists $(X_i,Y_i) \in P$ for each i such that $A_i \subseteq \mu(X_i,Y_i)$, and therefore $A_1 \cup A_2 \cup A_3 \subseteq \mu(X_1,Y_1) \cup \mu(X_2,Y_2) \cup \mu(X_3,Y_3) \neq V$ by (K2). To obtain (T4), suppose that $V \setminus \{v\} \in \mathcal{T}$. Then, there exists $(X,Y) \in P$ such that $V \setminus \{v\} \subseteq \mu(X,Y)$. Hence $\mu(X,Y) = V$ or $\mu(X,Y) = V \setminus \{v\}$, but we obtain a contradiction because of (K1) and (K3).

By the result of the previous theorem, we can provide a f-tangle kit as a certificate that branch-width is larger than k. In the following theorem we show that the size of its description is in a polynomial in $|V|$ and this certificate can be checked in time a polynomial in $|V|$ for fixed k.

THEOREM 3.3. *Let f be a connectivity function on subsets of V having branch-width larger than k. We assume that f is given by an oracle. For fixed k, there is a certificate that f has branch-width larger than k, of size at most a polynomial in $|V|$, that can be checked in time a polynomial in $|V|$.*

Proof. By Theorem 3.2, it is enough to provide a f-tangle kit (P,μ) of order $k+1$ to our algorithm as a certificate that branch-width of f is larger than k. Since $|P| \leq \sum_{i=0}^{k} \binom{|V|}{i}^2$, a description of (P,μ) has polynomial size in $|V|$.

Now we describe a polynomial-time algorithm that check that the certificate is valid, that is to decide whether (P,μ) satisfies its three axioms (K1), (K2), and (K3). By using submodular function minimization algorithms such as [7] or [3], we can calculate $f_{\min}$ in time a polynomial in $|V|$. So it is clear that those axioms can be checked in time a polynomial in $|V|$.

Suppose that we can calculate f by using an input of size in a polynomial in $|V|$ in polynomial time. By the previous theorem, we conclude that deciding whether the branch-width is at most k for fixed k is in NP∩coNP. But, it is still open whether it is in P. We conjecture that this is true.

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