Rank-width and Well-quasi-ordering

Sang-il Oum
Program in Applied & Computational Math.
Princeton Univ.

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Introduction

- Cut-rank function
- Rank-decomposition and Rank-width
- Clique-width
- Well-quasi-ordering

Cut-Rank Function

- G: graph.
- (A, B): partition of $V(G)$.

Let $M_A^B(G) = (m_{ij})_{i\in A, j\in B}$ be a $A \times B$ matrix over $\mathrm{GF}(2)$ such that

$$m_{ij} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j \\ 0 & \text{otherwise.} \end{cases}$$

Def: Cut-rank $\mathbf{cutrk}_G(A) = \mathbf{rank}(M_A^B(G))$.
Prop. cutrk_G is symmetric submodular, i.e.

$$\mathrm{cutrk}_G(X) + \mathrm{cutrk}_G(Y) \geq \mathrm{cutrk}_G(X \cap Y) + \mathrm{cutrk}_G(X \cup Y)$$
$$\mathrm{cutrk}_G(X) = \mathrm{cutrk}_G(V(G) \setminus X)$$

Rank-decomposition and Rank-width

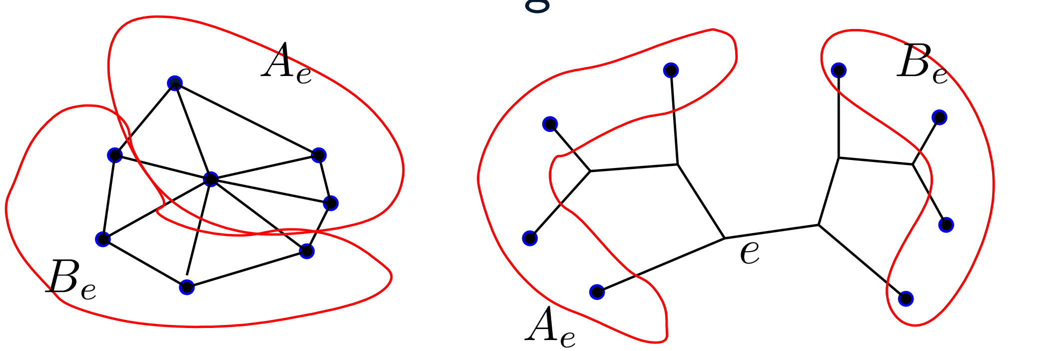
Def. • **Rank-decomposition** of G: (T, L). Cubic tree T, bijection $L : V \to \{x : x$ is a leaf of $T\}$.

• **Width** of (T, L):

$$\max_{e \in T} \operatorname{cutrk}_G(A_e)$$

where (A_e, B_e) is a partition of $V(G)$ induced by $e \in T$.

g



G Rank-decomposition of G

$$\operatorname{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

• **Rank-width** of G, denoted by $\operatorname{rwd}(G)$: minimum width over all possible rank-decompositions of G [Oum and Seymour, 2004]

Rank-width and Clique-width

- Clique-width: defined by [Courcelle and Olariu, 2000]
- (Rank-width and Clique-width are compatible)[Oum and Seymour, 2004]

$$\text{rank-width} \le \text{clique-width} \le 2^{\text{rank-width}+1} - 1$$

- Many NP-hard problems are solvable in polynomial time, if an input is restricted to graphs of bounded clique-width.

Let C be a set of graphs. We ask; "$\exists$ an alogrithm that, for every ??? formula φ, answers whether there exists $G \in C$ such that $\varphi(G)$ is true".

- (Seese's conjecture [Seese, 1991]) Bounded clique-width $\Leftrightarrow$ every MSOL formula on graphs is decidable on C. (open)
- ([Courcelle and Oum, 2004]) Bounded clique-width $\Leftrightarrow$ every MSOL formula with $Even(X)$ predicate on graphs is decidable on C.

Well-quasi-ordering

- $\leq$ is a quasi-ordering if reflexive ($a \leq a$) and transitive ($a \leq b, b \leq c \Rightarrow a \leq c$).
- A quasi-ordering $\leq$ on X is a **well-quasi-ordering** if for every infinite sequence $x_1, x_2, \ldots$ in X,

$$\exists i < j \text{ such that } x_i \leq x_j.$$

  In other words, X is **well-quasi-ordered** by $\leq$.
  Equivalently, every infinite sequence in X contains an infinite nondecreasing subsequence.
- Examples:(well-quasi-ordered) A set of positive integers with $\leq$. Any finite set. Finite trees with graph minor (Kruskal's theorem)
- Examples: (not well-quasi-ordered) A set of integers with $\leq$.

Graphs of Bounded Rank-width are well-quasi-ordered

WANTED: an appropriate quasi-ordering on graphs

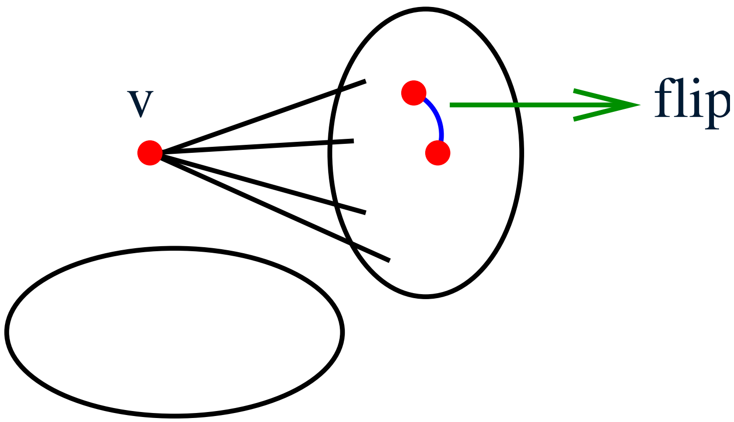
Induced Subgraph Relation is not enough

- Say $G_1 \leq G_2$ if G_1 is isomorphic to an induced subgraph of G_2.
- C_n: a cycle of length n.
- Consider $X = \{C_3, C_4, C_5, \ldots\}$.
- X has bounded rank-width (at most 4).
- no C_i is an induced subgraph of C_j $(i \neq j)$.

Note that if H is an induced subgraph of G, then
clique-width of $H \leq$ clique-width of G,
rank-width of $H \leq$ rank-width of G.

It would be nice if a set of graphs of bounded rank-width is **closed** under $\leq$. (So the graph minor is not appropriate!)

Local Complementation & Vertex-Minor



$G * v$

- $G * v$ and G have the same cut-rank function.
- G is **locally equivalent** to H if $H = G * v_1 * v_2 * \cdots v_k$.
- Call H is a **vertex-minor** of G, if H can be obtained by a sequence of local complementations and vertex deletions.

- $G * v$ and G have the same rank-width.
- Therefore, if H is a vertex-minor of G, then

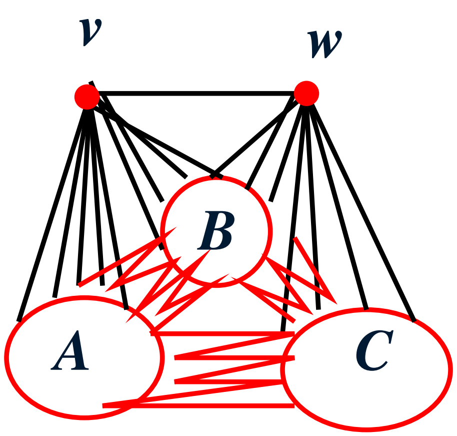
rank-width of $H \leq$ rank-width of G.

Statement of our thm

Thm. If $\{G_1, G_2, \ldots\}$ is an infinite sequence of graphs of rank-width $\leq k$, then there exists $i < j$ such that
G_i is **isomorphic** to a **vertex-minor** of G_j.

In fact, we prove a *stronger* theorem.

Thm. If $\{G_1, G_2, \ldots\}$ is an infinite sequence of graphs of rank-width $\leq k$, then there exists $i < j$ such that
G_i is isomorphic to a **pivot-minor** of G_j.



For an edge uv of G, the **pivoting** uv is an operation $G \wedge uv = G * u * v * u$.

H is a **pivot-minor** of G if
H is obtained from G by applying a sequence of pivoting and vertex deletions.

Tools

- **Isotropic system** [Bouchet, 1987] and Scraps
- **Extension of Menger's theorem** on scraps
- **If rank-width of G is n, then there is a linked rank-decompositon of width n.** [Geelen et al., 2002] cf. [Thomas, 1990]
  For any e, f in the rank-decomposition T, any vertex partition separating e, f has cut-rank $\geq$ min cut-rank of an edge in the path from e to f in T.
- **Robertson and Seymour's "Lemma on trees"** [Robertson and Seymour, 1990]

Binary matroids and wqo

Thm (Geelen, Gerards, Whilttle [Geelen et al., 2002]). If $\{M_1, M_2, \ldots\}$ is a sequence of binary matroids of branch-width $\leq k$,
then there exists $i < j$ such that M_i is **isomorphic** to a **minor** of M_j.

Tools

- "**Configuration**"
- Extension of Menger's theorem on matroids
- If branch-width of M is n, then there is a **linked** branch-decompositon of width n.
  For any e, f in the branch-decomposition T, any vertex partition separating e, f has connectivity $\geq$ min connectivity of an edge in the path from e to f in T.
- Robertson and Seymour's "Lemma on trees"

We generalize this theorem and mimic their proof.

Our thm implies GGW for binary matroids

1. For each M_i, pick a base B_i and construct a bipartite graph $G_i = Bip(M_i, B_i)$. Branch-width of M_i = Rank-width of G_i +1.
2. Fact: If H is a pivot-minor of G_i, then there exists a binary matroid M and its base B such that $H = Bip(M, B)$ and M is a minor of M_i.
3. [Seymour, 1988] If two binary matroids M, M' have the same connectivity function, then $M = M'$ or $M = M'^*$.
   If $Bip(M_i, B_i)$ is a vertex-minor of $Bip(M_j, B_j)$ and M_i is connected, then M_i is a minor of M_j or M_j^*.
4. **Connected** binary matroids of bounded branch-width is wqo.
   $\exists i < j < k$ such that $Bip(M_i, B_i)$ is isomorphic to a pivot-minor of $Bip(M_j, B_j)$ and $Bip(M_j, B_j)$ is isomorphic to a pivot-minor of $Bip(M_k, B_k)$.
   M_j is a minor of M_k or M_i is a minor of M_j or M_k.
5. Apply Higman's lemma to binary matroids.

Graph and Isotropic system

We introduce the notion of isotropic systems, defined by [Bouchet, 1987]. The minor of isotropic system is related to the vertex-minor of graphs. The $\alpha\beta$-minor of isotropic system is related to the pivot-minor of graphs.

Isotropic system

1. Let $K = \{0, \alpha, \beta, \gamma\}$ be a vector space over $\mathrm{GF}(2)$ with $\alpha + \beta + \gamma = 0$.
2. Let $\langle x, y\rangle$ be a bilinear form over K. It's uniquely determined; $\boldsymbol{\langle x, y\rangle = 1}$ if $\boldsymbol{0 \neq x \neq y \neq 0}$, $\langle x, y\rangle = 0$ otherwise.
3. K^V: set of functions from V to K. Vector space.
4. For $x, y \in K^V$, let $\boldsymbol{\langle x, y\rangle = \sum_{v\in V} \langle x(v), y(v)\rangle \in \mathrm{GF}(2)}$. This is a bilinear form.
5. A subspace L is called **totally isotropic**, if $\langle x, y\rangle = 0$ for all $x, y \in L$.

Note: $\dim(L) + \dim(L^\perp) = \dim(K^V) = 2|V|$. If L is totally isotropic, $L \subseteq L^\perp$.

Def ([Bouchet, 1987]). A pair $S = (V, L)$ is called **isotropic system** if

- V is a finite set and
- L is a totally isotropic subspace of K^V such that $\boldsymbol{\dim(L) = |V|}$.

Graph $\Rightarrow$ Isotropic system

For $x \in K^V$ and $P \subseteq V$, $x[P] \in K^V$ such that

$$x[P](v) = \begin{cases} x(v) & \text{if } v \in P \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a graph and $n(v)$ be the set of neighbors of v.

Let $a, b \in K^V$ such that $a(v), b(v) \neq 0$ for all v and $a(v) \neq b(v)$.

L is a vector space spanned by $\boldsymbol{\{a[n(v)] + b[\{v\}] : v \in V\}}$.
Then, $S = (V, L)$ is an isotropic system.

We call (G, a, b) the **graphic presentation** of S.

Isotropic System $\Rightarrow$ Graph

$a \in K^V$ is called **Eulerian vector** of $S = (V, L)$, if $a(v) \neq 0$ for all $v \in V$ and $a[P] \notin L$ for all $\emptyset \neq P \subseteq V$.

[Bouchet, 1988] showed

1. There exists an Eulerian vector for any isotropic system.
2. Let a be an Eulerian vector of $S = (V, L)$. For each v, there exists a **unique** vector $b_v \in L$ such that $b_v(v) \neq 0$ for all $v \in V$ and $b_v(w) = 0$ or $a(w)$ for all $w \neq v$.
   $\{b_v : v \in V\}$ is called the **fundamental basis** of S.

The **fundamental graph** of S is a graph (V, E) where

v, w are adjacent iff $b_v(w) \neq 0$.

By $\langle b_v(w), b_w(v) \rangle = 0$, $b_v(w) \neq 0$ iff $b_w(v) \neq 0$.

Local Complementation and Isotropic system

Let G be a graph. Let $\boldsymbol{c_v = a[n_G(v)] + b[\{v\}]}$.

Consider $G' = G * x$. Let $a' = a + b[\{x\}]$ and $b' = a[n_G(x)] + b$.

$$\boldsymbol{c'_v = a'[n_{G'}(v)] + b'[\{v\}] = \begin{cases} c_v + c_x & \textbf{if } v \sim x, \\ c_v & \textbf{otherwise.} \end{cases}}$$

Let L' be a vector space spanned by $\{c'_v\}$. Then, $L' = L$.

Local complementation of graphs doesnot change the associated isotropic system.

Minor

1. For $X \subseteq V$, $p_X : K^V \to K^X$ is a canonical projection such that $(p_X(x))(v) = x(v)$ for $v \in X$.
2. For a subspace L of K^V and $v \in V$, $a \in K - \{0\}$,

$$L|_a^v = \{p_{V-\{v\}}(x) : x \in L, \textbf{x(v)=0} \;\; \textbf{or} \;\; \textbf{a}\} \subseteq K^{V-\{v\}}.$$

For $a \in K - \{0\}$, $S|_a^v = (V - \{v\}, L|_a^v)$ is called an **elementary minor** of S.

S' is a **minor** of S if $S' = S|_{a_1}^{v_1}|_{a_2}^{v_2} \cdots |_{a_k}^{v_k}$ for some v_i, a_i.

S' is an **$\alpha\beta$-minor** of S if $S' = S|_{a_1}^{v_1}|_{a_2}^{v_2} \cdots |_{a_k}^{v_k}$ for some v_i, $a_i \in \{\alpha, \beta\}$.

Minor and Vertex-Minor

Thm (*[Bouchet, 1988]*). Let G be the fundamental graph of S.

Let H be the fundamental graph of $S|_x^v$.

Then, H is localley equivalent to one of $G \setminus v$, $G * v \setminus v$, or $G \wedge vw \setminus v$.

Cor. If S' is a minor of S, then the fundamental graph of S' is a vertex-minor of the fundamental graph of S.

$\alpha\beta$-Minor and Pivot-Minor

Thm. Let (G, a, b) be the graphic presentation of S such that $a(v), b(v) \in \{\alpha, \beta\}$ for all $v \in V(G)$.

Let (H, a', b') be the graphic presentation of S' such that $a'(v), b'(v) \in \{\alpha, \beta\}$ for all $v \in V(H)$.

If S' is an **$\alpha\beta$-minor** of S, then H is a **pivot-minor** of G.

“Actual” Main Theorem

We state the theorem written in the language of isotropic system. The proof heavily relies on

- combinatorial lemmas on vector space over $\mathrm{GF}(2)$ with form $\langle\,,\,\rangle$,
- isotropic system (or “scraps”),

Isotropic system and wqo

- Connectivity $\lambda_S(X) = |X| - \dim(L|_{\subseteq X}) = \text{CUT-RANK}_G(X)$.
- Branch-decomposition and branch-width of isotropic systems.
- $S_1 = (V_1, L_1)$ is **simply isomorphic** to $S = (V, L)$ if there is a bijectioin $\mu : V_1 \to V$ such that for any $x \in K^V$,

$$x \in L \text{ if and only if } x \cdot \mu \in L_1.$$

We prove the following.

Thm. If $\{S_1, S_2, \ldots\}$ is an infinite sequence of isotropic systems **of bounded branch-width**, then there exists $i < j$ such that S_i is simply isomorphic to an $\alpha\beta$-minor of S_j.

This implies our theorem about graphs and pivot-minor.

Scrap

$P = (V, L, B)$ is a **scrap** if V is a finite set and

- L is a totally isotropic subspace of K^V,
- B is an **ordered** set (sequence) and a **basis of $L^\perp/L$**.

$|B| = \dim(L^\perp/L) = (2|V| - \dim(L)) - \dim(L) = 2(|V| - \dim(L))$. If $B = \emptyset$, then (V, L) is an isotropic system.

$P_1 = (X, L', B')$ is a **minor** of P if $X = V \setminus \{v_1, v_2, \ldots, v_k\}$, $L' = L|_{x_1}^{v_1}|_{x_2}^{v_2} \cdots |_{x_k}^{v_k}$, and $|B'| = |B|$ and B' is obtained naturally from B by $\cdots$.

$P_1 = (X, L', B')$ is a **$\alpha\beta$-minor** of P if $X = V \setminus \{v_1, v_2, \ldots, v_k\}$, $L' = L|_{x_1}^{v_1}|_{x_2}^{v_2} \cdots |_{x_k}^{v_k}$ with $x_i \in \{\alpha, \beta\}$, and $|B'| = |B|$ and B' is obtained naturally from B by $\cdots$.

Very Rough Sketch of Proof

Suppose $\{S_1, S_2, \ldots\}$ is not well-quasi-ordered by $\alpha\beta$-minor relation.

Let F be an infinite forest such that each component is the **linked** branch-decomposition of S_i. We attach the root vertex to each component. For an edge e, let $l(e)$, $r(e)$ be the left/right child edge incident to e. We assign a **scrap** to each edge of F and define a relation $\leq$ on the set of edges of F. We make a scrap of e is a **sum** of scraps of $l(e)$ and $r(e)$.

By applying lemma on trees, we get a sequence e_0, e_1, ... of edges such that $\{e_0, e_1, \ldots\}$ is an antichain and $l(e_0) \leq l(e_1) \leq l(e_2) \leq \cdots$ and $r(e_0) \leq r(e_1) \leq r(e_2) \leq \cdots$.

The number of ways to **sum** 2 scraps is finite $\Rightarrow \exists i < j$, $e_i \leq e_j$. Contradiction.

Many (strange-looking?) lemmas

- $(L|_x^v)^\perp = L^\perp|_x^v$.
- If $X \subseteq V$, then $(L|_{\subseteq X})^\perp = L^\perp|_X$.
- $\dim(L|_x^v) = \begin{cases} \dim(L) & \text{if } \delta_x^v \in L^\perp \setminus L \\ \dim(L) - 1 & \text{otherwise.} \end{cases}$
- (Extension of Menger's theorem) Let $P = (V, L, B)$ be a scrap and $X \subseteq V$. If $\lambda(P) = \lambda(L|_{\subseteq X}) = \min_{X \subseteq Z \subseteq V} \lambda(L|_{\subseteq Z})$, then there is an ordered set B' such that $Q = (X, L|_{\subseteq X}, B')$ is a scrap and an $\alpha\beta$-minor of P.

Sum and Connection type

- "sum" of scraps
  $P = (V, L, B)$ is a **sum** of $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ if $V_1 \cap V_2 = \emptyset$ and $V = V_1 \cup V_2$.
  The number of distinct sums of P_1 and P_2 are finite up to simple isomorphisms (by "connection type" lemma).
- A **connection type** $C(P, P_1, P_2)$ determines P if P_1 and P_2 are given. Roughly speaking, it specifies how B and L are made from B_1 and B_2.
- The number of connection type is finite if $\lambda(P) = |V| - \dim(L)$ is bounded.
- If P_i is an ($\alpha\beta$-)minor of Q_i for $i = 1, 2$ and
  P is the sum of P_1 and P_2 and
  Q is the sum of Q_1 and Q_2.
  If $C(P, P_1, P_2) = C(Q, Q_1, Q_2)$, then P is an ($\alpha\beta$-)minor of Q.

Excluded vertex-minors for rank-width$\leq k$

G is an **excluded vertex-minor** for a class of graphs of rank-width$\leq k$ if

- Rank-width of $G > k$
- Every proper vertex-minor of G has rank-width$\leq k$.

Cor. For fixed k, there are **only finitely many excluded vertex-minors** for a class of graphs of rank-width $\leq k$.

Proof. An excluded vertex-minor has rank-width $k+1$. Let E be the set of excluded vertex-minors. E is well-quasi-ordered by the vertex-minor relation. But, no excluded vertex-minor contains another. So, E is finite. $\square$

Note: The above corollary has an elementary proof.[Oum, 2004]

Cor. For fixed k, **"Is rank-width$\leq$ k?"** is **NP$\cap$ coNP**.

In fact, this is in P. [Courcelle and Oum, 2004]

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