# Rank-width, Clique-width and Vertex-minors

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October 1, 2004

# Clique-width

**Def. 1.** [Courcelle and Olariu, 2000]

***k*-expression**: expression on vertex-labeled graphs with labels $\{1, 2, \cdots, k\}$ using the following 4 operations

| | |
|---|---|
| $G_1 \oplus G_2$ | disjoint union of $G_1$ and $G_2$ |
| $\eta_{i,j}(G)$ | add edges $uv$ for all $u$ and $v$ such that $lab(u) = i$, $lab(v) = j$ $(i \neq j)$ |
| $\rho_{i \to j}(G)$ | relabel all vertices of label $i$ into label $j$ |
| $\cdot_i$ | create a graph with one vertex with label $i$ |

**Clique-width** of $G$, denoted by $cwd(G)$:
minimum $k$ such that $G$ can be expressed by $k$-expression (after forgetting the labels)

# Constructing $k$-expressions

Suppose our input graphs are known to have clique-width $\leq 10$, but **inputs are given by its adjacency list**. How to constuct a 10-expression of an input graph?

It's open for $k > 3$ whether there exists a poly-time algorithm to find a $k$-expression assuming $cwd(G) \leq k$.
$k = 3$: [Corneil et al., 2000]
$k = 2$: [Corneil et al., 1985]

**Any algorithms that guarantee to find a $f(k)$-expression** also make algorithms based on $k$-expressions run in poly time, because $f(k)$ is independent of $n$.

# Definitions

We introduce a new width-parameter of $G$,

1. compatible with Clique-width,

2. similar to branch-width.

(Joint work with Paul Seymour)

# Cut-Rank Function

- $G$: graph.
- $(A, B)$: partition of $V(G)$.

Let $M_A^B(G) = (m_{ij})_{i \in A, j \in B}$ be a $A \times B$ matrix over $\mathbb{Z}_2$ such that

$$m_{ij} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j \\ 0 & \text{otherwise.} \end{cases}$$

Def: Cut-rank $\mathbf{cutrk}_G(A) = \mathbf{rank}(M_A^B(G))$.
*Prop.* 1. $\mathrm{cutrk}_G$ is symmetric submodular, i.e.

$$\mathrm{cutrk}_G(X) + \mathrm{cutrk}_G(Y) \geq \mathrm{cutrk}_G(X \cap Y) + \mathrm{cutrk}_G(X \cup Y)$$

$$\mathrm{cutrk}_G(X) = \mathrm{cutrk}_G(V(G) \setminus X)$$

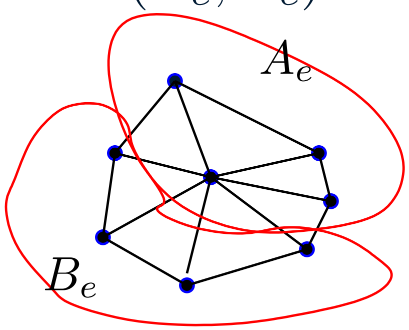
# Rank-decomposition and Rank-width

**Def. 2.** • **Rank-decomposition** of $G$: cubic tree $T$ with a bijection $L : V \longrightarrow \{x : x \text{ is a leaf of } T\}$.

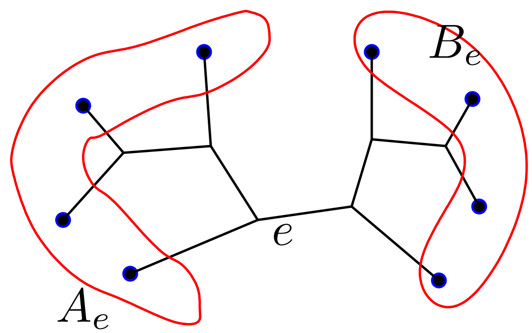
• **Width** of $(T, L)$:

$$\max_{e \in T} \operatorname{cutrk}_G(A_e)$$

where $(A_e, B_e)$ is a partition of $V(G)$ induced by $e \in T$.




$G$




Rank-decomposition of $G$

$$\operatorname{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

• **Rank-width** of $G$, denoted by $\operatorname{rwd}(G)$: minimum width over all possible rank-decompositions of $G$

# Rank-width and Clique-width

*Prop.* 2. (Rank-width and Clique-width are compatible)

$$\text{rank-width} \leq \text{clique-width} \leq 2^{\text{rank-width}+1} - 1$$

**Proof.** (Idea) If $M$ has at most $k$ distinct rows, then $\mathrm{rank}(M) \leq k$. Conversely, $\mathrm{rank}(M) \leq k$ implies $M$ has at most $2^k$ distinct rows/columns, if $M$ is a 0-1 matrix. $\square$

## Algorithms

- Converting rank-decomposition of width $\leq k$ into $(2^{k+1}-1)$-expression: $O(4^k n^2)$ time.
- Converting $k$-expression into rank-decomposition of width $\leq k$: $O(n)$ time.

# Vertex-minor of a graph

- If a graph $H$ is a minor of $G$, then
  tree-width of $H \leq$ tree-width of $G$.
- If a matroid $N$ is a minor of $M$, then
  branch-width of $N \leq$ branch-width of $M$.
- If a graph $H$ is a **vertex-minor** of $G$, then
  rank-width of $H \leq$ rank-width of $G$.

# Induced Subgraph Relation is not enough

From Robertson and Seymour:

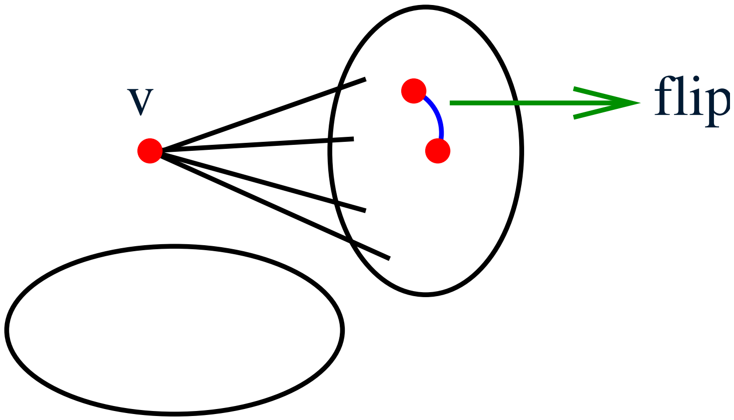
- For each $k$, there is a finite list of graphs $G_1$, $G_2$, $\cdots$, $G_m$ such that **clique-width**$(G) \leq k$ iff $\forall i$, $G_i$ is not isomorphic to any **minor** of $G$.

Suppose we have a finite list of graphs $G_1$, $G_2$, $\cdots$, $G_m$ such that **clique-width**$(G) \leq 3$ iff $\forall i$, $G_i$ is not isomorphic to any **induced subgraph** of $G$.

- $C_n$ (cycle of $n$ vertices) has clique-width 4 if $n > 6$.
- Every proper induced subgraph of $C_n$ is a union of vertex-disjoint paths. So, its clique-width is at most 3.
- Therefore, $C_n$ should be in the list.

Contradiction.

# Local Complementation & Vertex-Minor




$G * v$

- $G * v$ and $G$ have the same cut-rank function.
- $G$ is **locally equivalent** to $H$ if $H = G * v_1 * v_2 * \cdots v_k$.
- Call $H$ is a **vertex-minor** of $G$, if $H$ can be obtained by a sequence of local complementations and vertex deletions.

- $G * v$ and $G$ have the same rank-width.
- Therefore, if $H$ is a vertex-minor of $G$, then

rank-width of $H \leq$ rank-width of $G$.

# Vertex-Minor is NOT new

| A. Bouchet defined | But, I prefer |
|---|---|
| I-reduction | vertex-minor |
| connectivity | cut-rank |

1. $O(n^3)$ algorithm to check whether $H$ is **locally equivalent** to $G$ [Bouchet, 1991]
2. $G$ is a **circle graphs** (an intersection graph of chords in the circle) iff no 5-wheel, 7-wheel, $K_{4,3} -$ (3-edge-matching) as a **vertex-minor**. [Bouchet, 1994]
3. $G$: bipartite. $G$ is a circle graph iff a matroid represented by $G$ is a cycle matroid of a planar graph. [de Fraysseix, 1981]

# Graphs and Isotropic system

We introduce the notion of isotropic systems, defined by [Bouchet, 1987]. The minor of isotropic system is related to the vertex-minor of graphs.

# Isotropic system

1. Let $K = \{0, \alpha, \beta, \gamma\}$ be a vector space over $\mathrm{GF}(2)$ with $\alpha + \beta + \gamma = 0$.
2. Let $\langle x, y\rangle$ be a bilinear form over $K$. It's uniquely determined; $\boldsymbol{\langle x, y\rangle = 1}$ if $\mathbf{0} \neq \boldsymbol{x} \neq \boldsymbol{y} \neq \mathbf{0}$, $\langle x, y\rangle = 0$ otherwise.
3. $K^V$: set of functions from $V$ to $K$. Vector space.
4. For $x, y \in K^V$, let $\boldsymbol{\langle x, y\rangle = \sum_{v\in V}\langle x(v), y(v)\rangle} \in \mathrm{GF}(\mathbf{2})$. This is a bilinear form.
5. A subspace $L$ is called **totally isotropic**, if $\langle x, y\rangle = 0$ for all $x, y \in L$.

Note: $\dim(L) + \dim(L^\perp) = \dim(K^V) = 2|V|$. If $L$ is totally isotropic, $L \subseteq L^\perp$.

**Def. 3 ([Bouchet, 1987]).** A pair $S = (V, L)$ is called **isotropic system** if

- $V$ is a finite set and
- $L$ is a totally isotropic subspace of $K^V$ such that $\boldsymbol{\dim(L) = |V|}$.

# Graph ⇒ Isotropic system

For $x \in K^V$ and $P \subseteq V$, $x[P] \in K^V$ such that

$$x[P](v) = \begin{cases} x(v) & \text{if } v \in P \\ 0 & \text{otherwise.} \end{cases}$$

Let $G$ be a graph and $n(v)$ be the set of neighbors of $v$.

Let $a, b \in K^V$ such that $a(v) \neq 0$ for all $v$ and $a(v) \neq b(v)$.

$L$ is a vector space spanned by $\boldsymbol{\{a[n(v)] + b[\{v\}] : v \in V\}}$.

Then, $S = (V, L)$ is an isotropic system.

# Isotropic System $\Rightarrow$ Graph

$a \in K^V$ is called **Eulerian vector** of $S = (V, L)$, if $a(v) \neq 0$ for all $v \in V$ and $a[P] \notin L$ for all $\emptyset \neq P \subseteq V$.

[Bouchet, 1988] showed

1. There exists an Eulerian vector for any isotropic system.
2. Let $a$ be an Eulerian vector of $S = (V, L)$. For each $v$, there exists a **unique** vector $b_v \in L$ such that $b_v(v) \neq 0$ for all $v \in V$ and $b_v(w) = 0$ or $a(w)$ for all $w \neq v$.
   $\{b_v : v \in V\}$ is called the **fundamental base** of $S$.

The **fundamental graph** of $S$ is a graph $(S, E)$ where

$$v,\ w \text{ are adjacent iff } b_v(w) \neq 0.$$

By $\langle b_v(w), b_w(v)\rangle = 0$, $b_v(w) \neq 0$ iff $b_w(v) \neq 0$.

# Local Complementation and Isotropic system

Let $G$ be a graph. Let $\boldsymbol{c_v = a[n_G(v)] + b[\{v\}]}$.

Consider $G' = G * x$. Let $a' = a + b[\{x\}]$ and $b' = a[n_G(x)] + b$.

$$\boldsymbol{c'_v = a'[n_{G'}(v)] + b'[\{v\}] = \begin{cases} c_v + c_x & \textbf{if } v \sim x, \\ c_v & \textbf{otherwise.} \end{cases}}$$

Let $L'$ be a vector space spanned by $\{c'_v\}$. Then, $L' = L$.

Local complementation of graphs doesnot change the associated isotropic system.

# Minor

1. For $X \subseteq V$, $p_X : K^V \to K^X$ is a canonical projection such that $(p_X(x))(v) = x(v)$ for $v \in X$.
2. For a subspace $L$ of $K^V$ and $v \in V$, $a \in K - \{0\}$,

$$L|_a^v = \{p_{V-\{v\}}(x) : x \in L, x(v) = 0 \text{ or } a\} \subseteq K^{V-\{v\}}.$$

For $a \in K - \{0\}$, $S|_a^v = (V - \{v\}, L|_a^v)$ is called an **elementary minor** of $S$.

$S'$ is a **minor** of $S$ if $S' = S|_{a_1}^{v_1}|_{a_2}^{v_2} \cdots |_{a_k}^{v_k}$ for some $v_i$, $a_i$.

# Minor and Vertex-Minor

*Thm.* 1 *([Bouchet, 1988])*. Let $G$ be the fundamental graph of $S$.

Let $H$ be the fundamental graph of $S|_x^v$.

Then, $H$ is localley equivalent to one of $G \setminus v$, $G * v \setminus v$, or $G * v * w * v \setminus v$, where $w$ is a neighbor of $v$.

*Cor.* 1. If $S'$ is a minor of $S$, then the fundamental graph of $S'$ is a vertex-minor of the fundamental graph of $S$.

# Bipartite graphs and binary matroids

If a graph $G$ is **bipartite** and $M$ is a binary matroid represented by an bipartite adjacency matrix of $G$,

**rank-width of $G$ = branch-width of $M - 1$.**

# Matroid

$\boldsymbol{M = (E, I)}$ is a **matroid** if

- $E$ is a finite set,
- $I$ is a collection of **independent** subsets of $E$, satisfying
  (I1) $\emptyset \in I$
  (I2) If $A \in I$ and $B \subseteq A$, then $B \in I$.
  (I3) For any $Z \subseteq E$, the maximal subsets of $Z$ in $I$ have the same size $\boldsymbol{= r(Z)}$, the **rank** of $Z$.

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We call $M$ a **binary matroid** if $\exists$ matrix $N$ over $\mathbb{Z}_2$ such that

- $E$: set of column vectors of $N$,
- $I = \{X \subseteq E : X$ is independent as vectors$\}$.

# Standard Representaion of a Binary Matroid

- Elementary row operations don't change the binary matroid.
- We may delete dependent rows.

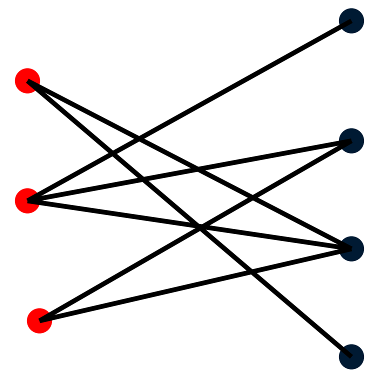
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 |

The **dual matroid** $M^*$:

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |

# Bipartite Graph from a Binary Matroid

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 |



$G$

Matroid operations

- **Deletion** $M \setminus e$ Delete the column.
- **Contraction** $M/e = (M^* \setminus e)^*$.

Matroid connectivity

$$\lambda_M(X) = r(X) + r(E(M) \setminus X) - r(E(M)) + 1$$

Let $G$ be bipartite, $M$ be a binary matroid given by $G$.

*Prop.* 3. $\boldsymbol{\lambda_M(X) = \mathrm{cutrk}_G(X) + 1}$.

*Cor.* 2. **branch-width of $M$ = Rank-width of $G$ +1.**

# Minor

A matroid $N$ is a **minor** of $M$ if $N$ is obtained by a sequence of contractions and deletions.

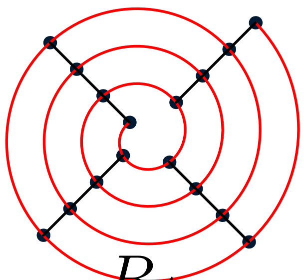
$Prop.$ 4. Let $H$, $G$ be a bipartite graph, $N$, $M$ be a binary matroid given by $H$, $G$ respectively. Then,

$$\boldsymbol{N} \textbf{ is a minor of } \boldsymbol{M} \Longrightarrow \boldsymbol{H} \textbf{ is a vertex-minor of } \boldsymbol{G}.$$

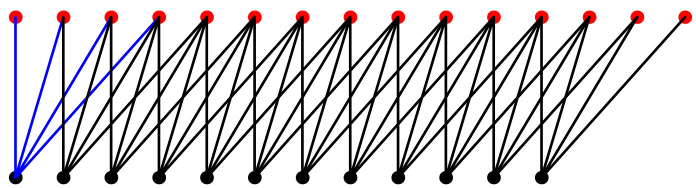
We can apply theorems about binary matroids.

# Grid theorem

*Thm.* 2. [Geelen et al., 2003] For $k$, there is large $l$ such that any binary matroid with branch-width $\geq l$ contains the cycle matroid of a $k \times k$ grid as a minor.



$R_4$



$S_4$

A $6k^2 \times 6k^2$ grid contains $R_k$ as a minor for large $k$ by [Robertson et al., 1994]. Thus, a binary matroid with huge branch-width contains the cycle matroid of $R_k$ as a minor for large $k$.

*Cor.* 3. For any $k$, there is large $l$ such that a **bipartite** graph with rank-width $\geq l$ contains $S_k$ as a **vertex-minor**.

# End of the first talk

Tomorrow, we will see

1. poly-time alogirithm to construct a rank-decomposition of width $\leq 3k+1$ if an input is a graph of rank-width at most $k$,
2. well-quasi-ordering of graphs of bounded rank-width under vertex-minor relation,
3. a poly-time algorithm to decide rank-width $\leq k$.

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