Approximating Rank-width and Clique-width Quickly

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Abstract. Rank-width is defined by Seymour and the author to investigate clique-width; they show that graphs have bounded rank-width if and only if they have bounded clique-width. It is known that many hard graph problems have polynomial-time algorithms for graphs of bounded clique-width, however, requiring a given decomposition corresponding to clique-width (*k-expression*); they remove this requirement by constructing an algorithm that either outputs a rank-decomposition of width at most $f(k)$ for some function f or confirms rank-width is larger than k in $O(|V|^9 \log |V|)$ time for an input graph $G = (V, E)$ and a fixed k. This can be reformulated in terms of clique-width as an algorithm that either outputs a $(2^{1+f(k)} - 1)$-expression or confirms clique-width is larger than k in $O(|V|^9 \log |V|)$ time for fixed k.
In this paper, we develop two separate algorithms of this kind with faster running time. We construct a $O(|V|^4)$-time algorithm with $f(k) = 3k+1$ by constructing a subroutine for the previous algorithm; we may now avoid using general submodular function minimization algorithms used by Seymour and the author. Another one is a $O(|V|^3)$-time algorithm with $f(k) = 24k$ by giving a reduction from graphs to binary matroids; then we use an approximation algorithm for matroid branch-width by Hliněný.

1 Preliminaries

In this paper, all graphs are simple, undirected, and finite.

Cut-rank functions. For a matrix $M = (m_{ij} : i \in C, j \in R)$ over a field F, if $X \subseteq R$ and $Y \subseteq C$, let $M[X, Y]$ denote the submatrix $(m_{ij} : i \in X, j \in Y)$. For a graph G, let $A(G)$ be its adjacency matrix over GF(2).

Definition 1. *Let G be a graph. For two disjoint subsets $X, Y \subseteq V(G)$, we define $\rho^*_G(X, Y) = \mathrm{rk}(A(G)[X, Y])$ where* rk *is the matrix rank function; and we define the* cut-rank *function ρ_G of G by letting $\rho_G(X) = \rho^*_G(X, V(G) \setminus X)$ for $X \subseteq V(G)$.*

Both ρ and ρ^* satisfy submodular inequalities.

Proposition 2 (O. and Seymour [1]). *Let G be a graph. Let X_1, Y_1, X_2, Y_2 be subsets of $V(G)$ such that $X_1 \cap Y_1 = X_2 \cap Y_2 = \emptyset$. Then,*

$$\rho^*_G(X_1, Y_1) + \rho^*_G(X_2, Y_2) \geq \rho^*_G(X_1 \cap X_2, Y_1 \cup Y_2) + \rho^*_G(X_1 \cup X_2, Y_1 \cap Y_2).$$

Moreover, if $X_1, X_2 \subseteq V(G)$*, then*

$$\rho_G(X_1) + \rho_G(X_2) \geq \rho_G(X_1 \cap X_2) + \rho_G(X_1 \cup X_2).$$

Rank-width. A *subcubic tree* is a tree with at least two vertices such that every vertex is incident with at most three edges. A *leaf* of a tree is a vertex incident with exactly one edge. A *rank-decomposition* of a graph $G = (V, E)$ is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijective function $\mathcal{L} : V \to \{t : t \text{ is a leaf of } T\}$. (If $|V| \leq 1$, then G admits no rank-decomposition.)

For an edge e of T, the connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a rank-decomposition $(T, \mathcal{L})$ is $\rho_G(\mathcal{L}^{-1}(X))$. The *width* of $(T, \mathcal{L})$ is the maximum width of all edges of T. The *rank-width* $\mathrm{rwd}(G)$ of G is the minimum width of a rank-decomposition of G. (If $|V| \leq 1$, we define $\mathrm{rwd}(G) = 0$.)

Let $\mathrm{cwd}(G)$ be the *clique-width* of a graph G. Clique-width is defined by Courcelle and Olariu [2]. In this paper, we do not need its definition if we just remember the following proposition.

Proposition 3 (O. and Seymour [1]). *For a graph* G*,* $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{\mathrm{rwd}(G)+1} - 1$*.*

Local complementation. For two sets A and B, let $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Definition 4. *Let* $G = (V, E)$ *be a graph and* $v \in V$*. The graph obtained by applying* local complementation *at* v *to* G *is*

$$G * v = (V, E \Delta \{xy : xv, yv \in E, x \neq y\}).$$

For an edge $uv \in E$*, the graph obtained by* pivoting uv *is defined by* $G \wedge uv = G * u * v * u$*. We say that* H *is* locally equivalent *to* G *if* G *can be obtained by applying a sequence of local complementations to* G*.*

A pivoting is well-defined because $G * u * v * u = G * v * u * v$ if u and v are adjacent [3]. The following observation is fundamental.

Proposition 5 (O. [3]). *Let* $G' = G * v$*. Then for every* $X \subseteq V(G)$*,*

$$\rho_G(X) = \rho_{G'}(X).$$

The following lemma will be used in Sect. 2.

Lemma 6 (O. [3]). *Let* G *be a graph and* $v \in V(G)$*. Suppose that* (X_1, X_2) *and* (Y_1, Y_2) *are partitions of* $V(G) \setminus \{v\}$*. If* w *is a neighbor of* v*, then*

$$\rho_{G \setminus v}(X_1) + \rho_{G \wedge vw \setminus v}(Y_1) \geq \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1.$$

Matroids. Since we will use matroids in Sect. 4, let us review matroid theory. For general matroid theory, we refer to Oxley's book [4]. We call $\mathcal{M} = (E, \mathcal{I})$ a *matroid* if E is a finite set and $\mathcal{I}$ is a collection of subsets of E, satisfying

(i) $\emptyset \in \mathcal{I}$
(ii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(iii) For every $Z \subseteq E$, maximal subsets of Z in $\mathcal{I}$ all have the same size $r(Z)$. We call $r(Z)$ the *rank* of Z.

An element of $\mathcal{I}$ is called *independent* in $\mathcal{M}$. We let $E(\mathcal{M}) = E$. A matroid $\mathcal{M} = (E, \mathcal{I})$ is *binary* if there exists a matrix N over GF(2) such that E is a set of column vectors of N and $\mathcal{I} = \{X \subseteq E : X \text{ is linearly independent}\}$. The *connectivity* function $\lambda_{\mathcal{M}}$ of $\mathcal{M}$ is $\lambda_{\mathcal{M}}(X) = r(X) + r(E \setminus X) - r(E) + 1$.

Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let $Bin(G, A, B)$ be the binary matroid on V, represented by the $A \times V$ matrix

$$\begin{pmatrix} I_A & A(G)[A, B] \end{pmatrix},$$

where I_A is the $A \times A$ identity matrix. If $\mathcal{M} = Bin(G, A, B)$, then G is called a *fundamental graph* of $\mathcal{M}$.

Branch-width. A *branch-decomposition* of a matroid $\mathcal{M}$ is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijective function $\mathcal{L} : E(\mathcal{M}) \to \{t : t \text{ is a leaf of } T\}$. (If $|E(\mathcal{M})| \leq 1$, then $\mathcal{M}$ admits no rank-decomposition.)

For an edge e of T, the connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a branch-decomposition $(T, \mathcal{L})$ is $\lambda_{\mathcal{M}}(\mathcal{L}^{-1}(X))$. The *width* of $(T, \mathcal{L})$ is the maximum width of all edges of T. The *branch-width* $\mathrm{bw}(\mathcal{M})$ of $\mathcal{M}$ is the minimum width of a branch-decomposition of $\mathcal{M}$. (If $|V| \leq 1$, we define $\mathrm{bw}(\mathcal{M}) = 1$.) Branch-width has been defined by Robertson and Seymour [5].

The following proposition links branch-width of binary matroids with rank-width of bipartite graphs.

Proposition 7 (O. [3]). *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then for every $X \subseteq V$, $\lambda_{\mathcal{M}}(X) = \rho_G(X) + 1$.*

Corollary 8 (O. [3]). *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then the branch-width of $\mathcal{M}$ is one more than the rank-width of G.*

2 Approximating Rank-width Quickly

In this section, we show that, for fixed k, there is a $O(n^4)$-time algorithm that, with a n-vertex graph, outputs a rank-decomposition of width at most $3k + 1$ or confirms that the input graph has rank-width larger than k. Oum and Seymour [1] use general submodular function minimization algorithms [6] to find Z minimizing the cut-rank function $\rho_G(Z)$ with $X \subseteq Z \subseteq V(G) \setminus Y$ for given disjoint subsets X, Y of $V(G)$ such that $|X|, |Y| \leq 3k$. If this can be done in time γ, then we obtain an $O(n(n^2 + \gamma))$-time algorithm to outputs a rank-decomposition of width at most $3k + 1$ or confirms that the input graph has rank-width larger than k. In [1], γ is $O(n^8 \log n)$, and therefore the $O(n^9 \log n)$-time algorithm is obtained.

To obtain a $O(n^4)$-time algorithm, we construct a direct combinatorial algorithm that minimizes the cut-rank function. Jim Geelen suggested the use of blocking sequences for this problem (private communication, 2005).

We first define *blocking sequences*, introduced by J. Geelen [7]. Let G be a graph and A, B be two disjoint subsets of $V(G)$. A sequence $v_1, v_2, \ldots, v_m$ of vertices in $V(G) \setminus (A \cup B)$ is called a *blocking sequence* for (A, B) in G if it satisfies the following:

(i) $\rho^*_G(A, B \cup \{v_1\}) > \rho^*_G(A, B)$.
(ii) $\rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+1}\}) > \rho^*_G(A, B)$ for all $i \in \{1, 2, \ldots, m-1\}$.
(iii) $\rho^*_G(A \cup \{v_m\}, B) > \rho^*_G(A, B)$.
(iv) No proper subsequence satisfies (i)—(iii).

The following proposition is used in most applications of blocking sequences.

Proposition 9. *Let G be a graph and A, B be two disjoint subsets of $V(G)$. The following are equivalent:*

(i) *There is no blocking sequence for (A, B) in G.*
(ii) *There exists Z such that $A \subseteq Z \subseteq V(G) \setminus B$ and $\rho_G(Z) = \rho^*_G(A, B)$.*

Proof. (i)→(ii): We assume that $a, b \notin V(G) \setminus (A \cup B)$ by relabeling. Let $k = \rho^*_G(A, B)$. We construct the *auxiliary digraph* $D = (\{a, b\} \cup (V(G) \setminus (A \cup B)), E)$ from G such that for $x, y \in V(G) \setminus (A \cup B)$,

i) $(a, x) \in E$ if $\rho^*_G(A, B \cup \{x\}) > k$,
ii) $(x, b) \in E$ if $\rho^*_G(A \cup \{x\}, B) > k$,
iii) $(x, y) \in E$ if $\rho^*_G(A \cup \{x\}, B \cup \{y\}) > k$.

Since there is no blocking sequence for (A, B) in G, there is no directed path from a to b in D. Let J be a set of vertices in $V(G) \setminus (A \cup B)$ having a directed path from a in D. We show that $Z = J \cup A$ satisfies $\rho_G(Z) = k$.

To prove this, we claim that $\rho^*_G(A \cup X, B \cup Y) = k$ for all $X \subseteq J$, $Y \subseteq V(G) \setminus (Z \cup B)$. We proceed by induction on $|X| + |Y|$. If $|X| \leq 1$ and $|Y| \leq 1$, then we have $\rho^*_G(A \cup X, B \cup Y) = k$ by the construction of J.

If $|X| > 1$, then for all $x \in X$ we have

$$\rho^*_G(A \cup X, B \cup Y) + \rho^*_G(A, B \cup Y) \leq \rho^*_G(A \cup (X \setminus \{x\}), B \cup Y) + \rho_G(A \cup \{x\}, B \cup Y) = 2k,$$

because $\rho^*_G(A \cup \{x\}, B \cup Y) = k$ by induction. So, $\rho^*_G(A \cup X, B \cup Y) = k$.

Similarly if $|Y| > 1$, then for all $y \in Y$ we have $\rho^*_G(A \cup X, B \cup Y) + \rho^*_G(A \cup X, B) \leq \rho^*_G(A \cup X, B \cup (Y \setminus \{y\})) + \rho_G(A \cup X, B \cup \{y\}) = 2k$, and therefore $\rho^*_G(A \cup X, B \cup Y) = k$.

(ii)→(i): Suppose that there is a blocking sequence $v_1, v_2, \ldots, v_m$. Then, $v_m \notin Z$ because $\rho^*_G(A \cup \{v_m\}, B) > \rho_G(Z)$. Similarly $v_1 \in Z$ because $\rho^*_G(A, B \cup \{v_1\}) > \rho_G(Z)$. Therefore there exists $i \in \{1, 2, \ldots, m-1\}$ such that $v_i \in Z$ but $v_{i+1} \notin Z$. But this is a contradiction, because $\rho_G(Z) < \rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+1}\}) \leq \rho^*_G(Z, V(G) \setminus Z) = \rho_G(Z)$. □

Lemma 10. *Let G be a graph (V, E) and A, B be two disjoint subsets of V such that $\rho^*_G(A, B) = k$ and $|A|, |B| \leq l$. Let $n = |V|$. There is a polynomial-time algorithm to either*

- *obtain a graph G' locally equivalent to G with $\rho^*_{G'}(A, B) > k$, or*
- *obtain a set Z such that $A \subseteq Z \subseteq V \setminus B$ and $\rho_G(Z) = k$.*

The running time of this algorithm is $O(n^3)$ if l is fixed or $O(n^4)$ if l is not fixed.

Proof. If there is no blocking sequence for (A, B) in G, then $\min_{A \subseteq Z \subseteq V \setminus B} \rho(Z) = k$ by Proposition 9. In this case, we obtain Z by finding a set of vertices reachable from A in the auxiliary graph.

Therefore, we may assume that there is a blocking sequence $v_1, v_2, \ldots, v_m$. We will find another graph G' locally equivalent to G such that $\mathrm{rk}_{G'}(A, B) > k$. Since $\mathrm{rk}_G(A \cup \{v_m\}, B) = k + 1$, there is a vertex $w \in B$ adjacent to v_m.

(1) We claim that $v_1, v_2, \ldots, v_{m-1}$ is a blocking sequence of (A, B) in $G \wedge v_m w$ if $m > 1$.

By applying Lemma 6 for $G[A \cup B \cup \{v_1, v_m\}]$, a subgraph of G induced on $A \cup B \cup \{v_1, v_m\}$, we have

$$\begin{aligned}\rho^*_{G \wedge v_m w}(A, B \cup \{v_1\}) &+ \rho^*_G(A \cup \{v_1\}, B) \\ &\geq \rho^*_G(A, B \cup \{v_1, v_m\}) + \rho^*_G(A \cup \{v_1, v_m\}, B) - 1.\end{aligned}$$

Since $\rho^*_G(A, B \cup \{v_1, v_m\} \geq \rho^*_G(A, B \cup \{v_1\}) \geq k + 1$, $\rho^*_G(A \cup \{v_1, v_m\}, B) \geq \rho^*_G(A \cup \{v_m\}, B) \geq k+1$, and $\rho^*_G(A \cup \{v_1\}, B) = k$, we obtain that $\rho^*_{G \wedge v_m w}(A, B \cup \{v_1\}) \geq k + 1$.

By applying the same inequality we obtain that

$$\begin{aligned}&\rho^*_{G \wedge v_m w}(A \cup \{v_i\}, B \cup \{v_{i+1}\}) + \rho^*_G(A \cup \{v_i, v_{i+1}\}, B) \\ &\quad \geq \rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+1}, v_m\}) + \rho^*_G(A \cup \{v_i, v_{i+1}, v_m\}, B) - 1 \geq 2k + 1\end{aligned}$$

for each $i \in \{1, 2, 3, \ldots, m-2\}$ and therefore $\rho^*_{G \wedge v_m w}(A \cup \{v_i\}, B \cup \{v_{i+1}\}) \geq k+1$.

Moreover, $\rho^*_{G \wedge v_m w}(A \cup \{v_{m-1}\}, B) + \rho^*_G(A \cup \{v_{m-1}\}, B) \geq \rho^*_G(A \cup \{v_{m-1}\}, B \cup \{v_m\}) + \rho^*_G(A \cup \{v_{m-1}, v_m\}, B) - 1 \geq 2k+1$ and therefore $\rho^*_{G \wedge v_m w}(A \cup \{v_{m-1}\}, B) \geq k + 1$.

We prove one lemma to be used later. If X and Y are disjoint subsets of V such that $A \subseteq X$, $B \subseteq Y$, $v_m \notin X \cup Y$ and $\rho^*_G(X, Y) = k$, then $\rho^*_{G \wedge v_m w}(X, Y) =$

$\rho^*_G(X, Y \cup \{v_m\})$ because

$$\begin{aligned}\rho^*_{G\wedge v_m w}(X,Y) + \rho^*_G(X,Y) &\geq \rho^*_G(X, Y\cup\{v_m\}) + \rho^*_G(X\cup\{v_m\}, Y) - 1\\ &\geq \rho^*_G(X, Y\cup\{v_m\}) + k = \rho^*_{G\wedge v_m w}(X, Y\cup\{v_m\}) + \rho^*_G(X,Y).\end{aligned}$$

By letting $X = A \cup \{v_{m-1}\}$ and $Y = B$, we obtain that $\rho^*_{G\wedge v_m w}(A \cup \{v_{m-1}\}, B) = \rho^*_G(A\cup\{v_{m-1}\}, B\cup\{v_m\}) \geq k+1$. We also obtain $\rho^*_{G\wedge v_m w}(A, B\cup \{v_i\}) = k$ for each $i > 1$ by letting $X = A$, $Y = B \cup \{v_i\}$. Similarly we obtain $\rho^*_{G\wedge v_m w}(A\cup\{v_i\}, B\cup\{v_j\}) = k$ for i, j such that $1 \leq i < i+1 < j \leq m-1$.

Therefore, $v_1, v_2, \ldots, v_{m-1}$ is a blocking sequence for (A, B) in $G \wedge v_m w$.

(2) If $m = 1$ then we obtain $\rho^*_{G\wedge v_1 w}(A, B) \geq k+1$, by applying the previous lemma with letting $X = A$ and $Y = B$.

(3) For each k, we claim that we can obtain another graph G' locally equivalent to G with $\rho^*_{G'}(A, B) > k$ or find Z satisfying $A \subset Z \subseteq V \setminus B$ and $\rho_G(Z) = k$.

If l is fixed, then we can test an adjacency in the auxiliary graph (defined in the proof of Proposition 9) in constant time by calculating rank of matrices of size no bigger than $(l+1)\times(l+1)$, and therefore it takes $O(n^2)$ time to construct the auxiliary digraph. If l is not fixed, then it takes $O(n^4)$ time to construct the auxiliary digraph for finding a blocking sequence. We first obtain the diagonalized matrix R obtained by applying elementary row operations to the matrix $M[A, B]$ in $O(n^3)$ time. For each vertex v not in $A \cup B$, we calculate the rank of $M[A\cup\{v\}, B]$ by using the stored matrix in $O(n^2)$ time. Similarly we calculate the rank of $M[A, B\cup\{v\}]$ by storing the matrix obtained by applying elementary column operations to $M[A, B]$. To check whether $\rho^*_G(A\cup\{x\}, B\cup\{y\}) > k$, it is enough to see when $\rho^*_G(A\cup\{x\}, B) = \rho^*_G(A, B\cup\{y\}) = k$. We first store the rows of the original matrices to each column of R and then we obtain the linear combination of rows of $M[A, B]$ giving $M[\{x\}, B]$. By the same linear combination, we check whether rows of $M[A, \{y\}]$ gives $M[\{x\}, \{y\}]$. It takes $O(n^2)$ time for each $x, y \in V \setminus (A \cup B)$ and therefore we construct the auxiliary digraph in $O(n^4)$ time (if l is not fixed).

To find a blocking sequence, it is enough to find a shortest path in this digraph and it takes $O(n^2)$ time. If there is no blocking sequence, then we find Z in $O(n^2)$ time by choosing all vertices reachable from A by a directed path.

We pick a neighbor of v_m in B and obtain $G \wedge v_m w$ in $O(n^2)$ time. By (1), $G \wedge v_m w$ has a blocking sequence $v_1, v_2, \ldots, v_{m-1}$ for (A, B). We apply this kind of pivoting m times so that in the new graph G' we have $\rho^*_{G'}(A, B) > k$. Since $m \leq n$, we obtain G' in $O(n^3)$ time. □

Theorem 11. *Let l be a fixed constant. Let G be a graph (V, E) and A, B be two disjoint subsets of V such that $|A|, |B| \leq l$. Then, there is a $O(|V|^3)$-time algorithm to find Z with $A \subseteq Z \subseteq V \setminus B$ having the minimum cut-rank.*

Proof. We apply the algorithm given by Lemma 10 until it finds a cut. We use the algorithm at most l times, and so the running time is at most $O(|V|^3)$. □

We state the following theorem for the sake of its own interest. We will not use this for the purpose of approximating rank-width since we have the previous theorem.

Theorem 12. *Let G be a graph (V, E) and A, B be two disjoint subsets of V. Then, there is a $O(|V|^5)$-time algorithm to find Z with $A \subseteq Z \subseteq V \setminus B$ having the minimum cut-rank.*

Proof. We apply the algorithm given by Lemma 10 until it finds a cut. We use the algorithm at most $|V|$ times, and so the running time is at most $O(|V|^5)$. $\square$

Combining with Oum and Seymour [1], we obtain the following.

Theorem 13. *For given k, there is an algorithm, for the input graph $G = (V, E)$, that either concludes that $\operatorname{rwd}(G) > k$ or outputs a rank-decomposition of G of width at most $3k + 1$; and its running time is $O(|V|^4)$.*

Since we can convert the rank-decomposition of width k to a $(2^{k+1}-1)$-expression (a decomposition related to clique-width) in $O(|V|^2)$ time [1], we obtain the following corollary.

Corollary 14. *For given k, there is an algorithm, for the input graph $G = (V, E)$, that either concludes that $\operatorname{cwd}(G) > k$ or outputs a $(2^{3k+2}-1)$-expression of G; and its running time is $O(|V|^4)$.*

3 Graphs to Bipartite Graphs

Courcelle [8] shows that Seese's conjecture [9] is true if and only if it is true for bipartite graphs by using a certain graph transformation B from graphs to bipartite graphs which we describe in the following lemma. He proves that there exist two functions f_1 and f_2 such that $f_1(\operatorname{rwd}(G)) \le \operatorname{rwd}(B(G)) \le f_2(\operatorname{rwd}(G))$, but does not have explicit constructions of f_1 and f_2. We give a concrete bound on rank-width. We will use this lemma in Sect. 4.

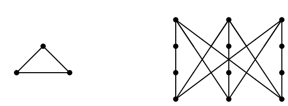

Fig. 1. K_3 and $B(K_3)$

Lemma 15. *Let $G = (V, E)$ be a graph. Let $B(G) = (V \times \{1, 2, 3, 4\}, E')$ be a bipartite graph obtained from G as follows:*

(i) *if $v \in V$ and $i \in \{1, 2, 3\}$, then (v, i) is adjacent to $(v, i + 1)$ in $B(G)$,*
(ii) *if $vw \in E$, then $(v, 1)$ is adjacent to $(w, 4)$ in $B(G)$.*

Then we have $\frac{1}{4}\operatorname{rwd}(G) \le \operatorname{rwd}(B(G)) \le \max(2\operatorname{rwd}(G), 1)$.

Proof. (1) Let us show that $\operatorname{rwd}(B(G)) \le \max(2\operatorname{rwd}(G), 1)$. If $\operatorname{rwd}(G) = 0$, then $\operatorname{rwd}(B(G)) = 1$. Now we may assume that $\operatorname{rwd}(G) > 0$ and we claim that $\operatorname{rwd}(B(G)) \le 2\operatorname{rwd}(G)$. Let $(T, \mathcal{L})$ be a rank-decomposition of G of width k. Let N be the set of leaves of T. Let T' be a tree such that $V(T') = (V(T) \times \{0\}) \cup (N \times \{1, 2, 3, 4, 12, 34\})$ and

(i) if $vw \in E(T)$, then $(v, 0)$ is adjacent to $(w, 0)$ in T',
(ii) for all $v \in N$, $(v, 12)$ is adjacent to both $(v, 1)$ and $(v, 2)$ in T',
(iii) for all $v \in N$, $(v, 34)$ is adjacent to both $(v, 3)$ and $(v, 4)$ in T',
(iv) for all $v \in N$, $(v, 0)$ is adjacent to both $(v, 12)$ and $(v, 34)$ in T'.

Informally speaking, we obtain T' from T by replacing each leaf with a rooted binary tree having four leaves. For each vertex (v, i) of $B(G)$, we define $\mathcal{L}'((v, i)) = (\mathcal{L}(v), i) \in V(T')$. Then $(T', \mathcal{L}')$ is a rank-decomposition of $B(G)$.

We claim that the width of $(T', \mathcal{L}')$ is at most $2k$.

For each edge $e = vw \in E(T)$, let (X, Y) be a partition of N induced by the connected components of $T \setminus e$. Then, the edge $(v, 0)(w, 0)$ of $E(T')$ induces a partition $(X \times \{1, 2, 3, 4\}, Y \times \{1, 2, 3, 4\})$ of $N \times \{1, 2, 3, 4\}$. We observe that $\mathcal{L}'^{-1}(X \times \{1, 2, 3, 4\}) = \mathcal{L}^{-1}(X) \times \{1, 2, 3, 4\}$. It is easy to see that

$$\rho_{B(G)}(\mathcal{L}'^{-1}(X \times \{1, 2, 3, 4\}) = 2\rho_G(\mathcal{L}^{-1}(X)) \le 2k.$$

We now consider remaining edges of T'. Each of them induces a partition (X, Y) of leaves of T' such that $|X| \le 2$ or $|Y| \le 2$. So, $\rho_{B(G)}(\mathcal{L}'^{-1}(X)) \le 2$. Therefore we obtain that the width of $(T', \mathcal{L}')$ is at most $2k$.

(2) We claim that $\operatorname{rwd}(G) \le 4\operatorname{rwd}(B(G))$. Let $(T, \mathcal{L})$ be a rank-decomposition of $B(G)$ of width k. Let e be an edge of T, and (X, Y) be a partition of leaves of T induced by connected components of $T \setminus e$.

For four subsets A_1, A_2, A_3, A_4 of V, we denote $A_1|A_2|A_3|A_4 = (A_1 \times \{1\}) \cup (A_2 \times \{2\}) \cup (A_3 \times \{3\}) \cup (A_4 \times \{4\})$ to simplify our notation. Let $\mathcal{L}^{-1}(X) = A_1|A_2|A_3|A_4$. Let $B_i = V \setminus A_i$ for $i \in \{1, 2, 3, 4\}$.

It is easy to observe, for each $i \in \{1, 2, 3\}$, that $\rho^*_{B(G)}((A_i \times \{i\}) \cup (A_{i+1} \times \{i+1\}), (B_i \times \{i\}) \cup (B_{i+1} \times \{i+1\}) = |A_i \cap B_{i+1}| + |B_i \cap A_{i+1}| = |A_i \Delta A_{i+1}|$. Since $\rho_{B(G)}(A_1|A_2|A_3|A_4) = \rho^*_{B(G)}(A_1|A_2|A_3|A_4, B_1|B_2|B_3|B_4) \le k$, we have, for each $i \in \{1, 2, 3\}$,

$$|A_i \Delta A_{i+1}| \le \rho_{B(G)}(A_1|A_2|A_3|A_4) \le k.$$

By adding these inequalities for all i, we obtain that $|A_1 \Delta A_4| \le 3k$.

We also observe that $\operatorname{rk}(M[A_4, B_1]) = \rho_{B(G)}(A_4 \times \{4\}, B_1 \times \{1\}) \le k$. Let M be an adjacency matrix of G. Then we have the following bound of $\rho_G(A_1)$:

$$\begin{aligned}\rho_G(A_1) = \operatorname{rk}(M[A_1, B_1]) &\le \operatorname{rk}(M[A_4 \cup (A_4 \Delta A_1), B_1]) \\ &\le \operatorname{rk}(M[A_4, B_1]) + \operatorname{rk}(M[A_4 \Delta A_1, B_1]) \le 4k.\end{aligned}$$

Let T' be the minimal subtree of T containing all leaves in $\mathcal{L}(V \times \{1\})$. Let $\mathcal{L}'(v) = \mathcal{L}((v, 1))$ for all vertices v of G. Then $(T', \mathcal{L}')$ is a rank-decomposition of G and its width is at most $4k$. $\square$

4 Approximating Rank-width More Quickly

In this section, we show another algorithm that approximate rank-width as in Sect. 2, but in $O(n^3)$ time with a worse approximation ratio. We take a different approach based on a simple observation in Sect. 3. We use the following algorithm for binary matroids developed by Hliněný [10].

Theorem 16 (Hliněný [10, Theorem 4.12]). *For fixed k, there is a $O(n^3)$-time algorithm that, for a given binary matroid with n elements, obtains a branch-decomposition of width at most $3k+1$ or confirms that the given matroid has branch-width larger than $k+1$. We assume that binary matroids are given by their matrix representations.*

This algorithm can be used to approximate rank-width of a bipartite graph G because we can run this algorithm for binary matroids having G as a fundamental graph. By Lemma 15, we obtain a bipartite graph $B(G)$ for each graph G such that $\frac{1}{4}\operatorname{rwd}(G) \le \operatorname{rwd}(B(G)) \le \max(2\operatorname{rwd}(G), 1)$. Moreover we can construct $B(G)$ in $O(n^2)$ time when $n = |V(G)|$ and transform the rank-decomposition of $B(G)$ of width m into rank-decomposition of G of width at most $4m$ in linear time by the proof of Lemma 15. Therefore, we obtain the following algorithm.

Corollary 17. *For fixed k, there is a $O(n^3)$-time algorithm that, for a given graph with n vertices, obtains a rank-decomposition of width at most $24k$ or confirms that the rank-width of the input graph is larger than k.*

Proof. Let $G = (V, E)$ be the input graph. We may assume that $E(G) \neq \emptyset$. First we construct $B(G)$ in $O(n^2)$ time. We run the algorithm of Theorem 16 with an input $\mathcal{M} = Bin(B(G), V \times \{1, 3\}, V \times \{2, 4\})$ and a constant $2k$.

If it confirms that branch-width of $\mathcal{M}$ is larger than $2k+1$, then rank-width of $B(G)$ is larger than $2k$, and therefore the rank-width of G is larger than k.

If it outputs the branch-decomposition of $\mathcal{M}$ of width at most $6k+1$, then the output is a rank-decomposition of $B(G)$ of width at most $6k$. This can be transformed into a rank-decomposition of G of width at most $24k$ in linear time by using an argument of Lemma 15. □

5 Discussions

Many applications of clique-width are polynomial-time algorithms to solve graph problems when inputs are restricted to graphs of bounded clique-width. Most of them ([11,12,13,14,15]) require k-expression of the input graph as an input to take an advantage of tree-structures (except Johnson [16]). But by using [1], we do not need k-expressions as an explicit input, because we can generate a $(2^{1+f(k)}-1)$-expression in polynomial time and provide it as an input. The result of this paper will make this preprocessing much faster.

In [17], Courcelle and the author show that there is a $O(|V|^9 \log |V|)$-time algorithm that recognizes graphs of rank-width at most k for an input graph $G = (V, E)$ and a fixed k; they use an approximation algorithm by Seymour and the author [1] as a first step, and it is the slowest part of their algorithm. By the result of this paper, we obtain the following.

Theorem 18. *For fixed k, there is a $O(n^3)$-time algorithm to check that the input graph with n vertices has rank-width at most k.*

But it is still open whether, for fixed k, we can construct a rank-decomposition of width at most k if there are any in polynomial time.

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